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A Generalization of Gorenstein Injective and Flat Modules

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Abstract: In this article, we introduce and study the concept of n-Gorenstein injective (resp., n-Gorenstein flat) modules as a nontrivial generalization of Gorenstein injective (resp., Gorenstein flat) modules. We investigate the properties of these modules in various ways. For example, we show that the class of n-Gorenstein injective (resp., n-Gorenstein flat) modules is closed under direct sums and direct products for $n \geq 2$. To this end, we first introduce and study the notions of n-injective modules and n-flat modules.

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1 Introduction

Let R be a ring. A left R-module N is called Gorenstein injective if there is a $\mathrm{Hom}(\mathcal{I}nj,\ -)$ exact exact sequence

$$\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$$

of injective left R-modules such that $N = \ker(E^0 \to E^1)$, where $\mathcal{I}nj$ stands for the class of all injective left R-modules (see [1]). A right R-module M is called Gorenstein flat if there is a $- \otimes \mathcal{I}nj$ exact exact sequence

$$\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

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of flat right R-modules such that M=ker$(F^0 \to F^1)$ (see [2]). Gorenstein injective and Gorenstein flat modules have been studied by many authors (see [1]–[5] and so on). These modules have nice properties when the ring in question is n-Gorenstein (a ring R is called n-Gorenstein if R is a left and right Noetherian ring with self-injective dimension at most n for an integer $n \geq 0$ on either side). For example, if R is n-Gorenstein and G is a Gorenstein injective left (right) R-module, then G^+ is a Gorenstein flat right (left) R-module; if R is Gorenstein, then arbitrary products and sums of Gorenstein flat modules are Gorenstein flat. However, these results are not true over general rings (see [6]).

The main purpose of this paper is to extend some Gorenstein homological properties over Noetherian rings or coherent rings to any ring. To this end, we introduce a generalization of Gorenstein injective (resp., Gorenstein flat) modules which forms a class closed under direct sums (resp., direct products) over any ring. These modules are called n-Gorenstein injective (resp., n-Gorenstein flat) modules. Therefore, although the direct sum (resp., direct product) of Gorenstein injective (resp., Gorenstein flat) modules need not be Gorenstein injective (resp., Gorenstein flat) in general, the direct sum (resp., direct product) of Gorenstein injective (resp., Gorenstein flat) modules is always 2-Gorenstein injective (resp., 2-Gorenstein flat). And an R-module M is -1-Gorenstein injective (-1-Gorenstein flat) if and only if M is Gorenstein injective (Gorenstein flat). We study the properties of n-Gorenstein injective modules and n-Gorenstein flat modules over any ring. For example, it is shown that a right R-module M is n-Gorenstein flat if and only if there is a $- \otimes n\mathcal{PI}$ exact exact sequence $0 \to M \to F^0 \to F^1 \cdots$ with each F^i n-flat and $\mathrm{Tor}_1(M,\ E) = 0$ for all n-presented injective left R-modules E and a left R-module M is n-Gorenstein injective if and only if there is a $\mathrm{Hom}(n\mathcal{PI},\ -)$ exact exact sequence $\cdots E^1 \to E^0 \to M \to 0$ with each E^i n-injective and $\mathrm{Ext}^1(E,\ M) = 0$ for all n-presented injective left R-modules E, where $n\mathcal{PI}$ denotes the class of all n-presented injective left R-modules. We prove that M^+ is n-Gorenstein flat for any n-Gorenstein injective left R-module M with $n \geq 2$ over any ring R. We also show that ${}_RR$ is n-injective if and only if every left R-module is n-Gorenstein injective.

Let us recall some known notions and facts needed in the sequel.

Let M and N be R-modules. $M^+ = \mathrm{Hom}_Z(M,\ Q/Z)$ denotes the character module of M. $\mathrm{Hom}(M,\ N)$ (resp., $\mathrm{Ext}^n(M,\ N)$) means $\mathrm{Hom}_R(M,\ N)$ (resp., $\mathrm{Ext}^n_R(M,\ N)$), and similarly $M \otimes N$ (resp., $\mathrm{Tor}_n(M,\ N)$) denotes $M \otimes_R N$ (resp., $\mathrm{Tor}^R_n(M,\ N)$) for an integer $n \geq 1$.

Let $\mathscr{C}$ be a class of R-modules and M an R-module. Recall that a homomorphism $\phi : C \to M$ is a $\mathscr{C}$-precover of M if $C \in \mathscr{C}$ and the abelian group homomorphism $\mathrm{Hom}(C',\ \phi) : \mathrm{Hom}(C',\ C) \to \mathrm{Hom}(C',\ M)$ is surjective for every $C' \in \mathscr{C}$ (see [7]). A $\mathscr{C}$-precover $\mathscr{C} : C \to M$ is said to be a $\mathscr{C}$-cover of M if every endomorphism $g : C \to C$ such that $\phi g = \phi$ is an isomorphism. Dually, we have the definitions of a $\mathscr{C}$-preenvelope and a $\mathscr{C}$-envelope. $\mathscr{C}$-covers ($\mathscr{C}$-envelopes) may not exist in general, but if they exist, they are unique up to isomorphism.

Let R be a ring and n a nonnegative integer. Following Costa[8], a left R-module is called