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Existence of Solution for Fractional Differential Problem with a Parameter

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Abstract: We apply the method of lower and upper solutions combined with monotone iterations to fractional differential problem with a parameter. The existence of minimal and maximal solutions is proved for the fractional differential problem with a parameter.

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1 Introduction

We consider the following fractional differential problem with a parameter:

$$\begin{cases} D^\alpha u = f(t,\, u(t),\, \lambda), & t \in [0, T], \\ u(0) = u_0, \quad G(u(T),\, \lambda) = 0, \end{cases} \tag{1.1}$$

where $0 < T < +\infty$, $\lambda \in \mathbf{R}$, $f \in C([0,\, T] \times \mathbf{R} \times \mathbf{R},\, \mathbf{R})$, $G \in C(\mathbf{R} \times \mathbf{R},\, \mathbf{R})$, $u_0 \in \mathbf{R}$, and D^α is Caputo fractional derivative of order $0 < \alpha < 1$ defined by

$$\begin{aligned} D^\alpha u(t) &= \frac{1}{\Gamma(1-\alpha)}\Big[\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-s)^{-\alpha} u(s)\mathrm{d}s - t^{-\alpha}u(0)\Big] \\ &= \frac{\mathrm{d}}{\mathrm{d}t} I^{1-\alpha}u(t) - \frac{u(0)}{\Gamma(1-\alpha)} t^{-\alpha}, \end{aligned}$$

where

$$\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha} u(s)\mathrm{d}s = I^{1-\alpha}u(t)$$

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is the Riemann-Liouville fractional integral of order $1 - \alpha$ (see [1]).

Integer order differential problem with a parameter has been studied for many years (see [2]). Differential equations of fractional order occur more frequently on different research areas and engineering, such as physics, chemistry, control of dynamical, etc. Recently, many authors pay attention to the existence result of solutions of initial value problem for fractional differential equations (see [3–15]). Motivated by [9–15], we consider the existence of the minimal and maximal solutions of (1.1), employing the classical proofs of differential equations-monotone iterative method.

Definition 1.1 *We say that a pair* $(u, \lambda) \in C^\alpha([0, T], \mathbf{R}) \times \mathbf{R}$ *is a solution of* (1.1) *if* (u, λ) *satisfies* (1.1), *where*

$$C^\alpha([0, T], \mathbf{R}) = \{u \in C([0,T]) : D^\alpha u(t) \in C([0, T])\}.$$

Lemma 1.1([1], Lemma 2.22) *If* $f(t) \in AC([0, T])$ *or* $f(t) \in C([0, T])$, *and* $0 < \alpha < 1$, *then*

$$I^\alpha D^\alpha f(t) = f(t) - f(0).$$

Lemma 1.2([1], Lemma 2.3) $I^p I^q f(t) = I^{p+q} f(t),\ f \in L([0, T]),\ p, q > 0.$

The following lemma is an existence result of solution for the linear initial value problem for a fractional differential equation, which is important for us to obtain the existence result of solution for (1.1).

Lemma 1.3([1], Theorem 4.3) *The linear initial value problem*

$$\begin{cases} D^\alpha u + du = q(t), & t \in (0, T], \\ u(0) = u_0, \end{cases} \tag{1.2}$$

where d *is a constant and* $q \in C([0, T] \times \mathbf{R})$, *has a unique solution* $u(t) \in C^\alpha([0, T], \mathbf{R})$, *and this solution is given by*

$$u(t) = u_0 E_{\alpha,1}(-dt^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-d(t-s)^\alpha) q(s)\mathrm{d}s, \tag{1.3}$$

where $E_{\alpha,\alpha}(-dt^\alpha)$ *is Mittag-Leffler function.*

In particular, when $d = 0$, *the initial value problem* (1.2) *has a solution*

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} q(s)\mathrm{d}s.$$

2 Main Result

In this section, we devote to considering the existence result of the minimal and maximal solutions of (1.1), by means of the monotone iterative method.

The following is the definition of the upper and lower solutions of (1.1).

Definition 2.1 *A pair* $(v, \mu) \in C^\alpha([0, T], \mathbf{R}) \times \mathbf{R}$ *is called a upper solution of* (1.1), *if it satisfies*

$$\begin{cases} D^\alpha v(t) \geq f(t, v(t), \mu), & t \in (0, T], \\ v(0) \geq u_0, \quad G(v(T), \mu) \leq 0. \end{cases}$$