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An Extension of Chebyshev's Maximum Principle to Several Variables

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Abstract: In this article, we generalize Chebyshev's maximum principle to several variables. Some analogous maximum formulae for the special integration functional are given. A sufficient condition of the existence of Chebyshev's maximum principle is also obtained.

Key words: cubature formula, orthogonal polynomial, Chebyshev's maximum principle, nonstandard Gaussian quadrature

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1 Introduction

Let ω be a nonnegative weight function on $(-\infty, \infty)$ for which

$$x^n\omega(x) \in L^1(-\infty, \infty), \qquad n = 0, 1, \cdots$$

We construct a sequence of orthonormal polynomials

$$p_n(\omega, x) = \gamma_n(\omega)x^n + \cdots, \qquad \gamma_n(\omega) > 0,$$

to satisfy

$$\int_{-\infty}^{\infty} \omega(x)p_n(\omega, x)p_m(\omega, x)\mathrm{d}x = \delta_{mn}.$$

It is well known that all zeros x_{kn} of $p_n(\omega)$ are real and simple. Let us denote by $X_n(\omega)$ the greatest zero of $p_n(\omega)$. The sequence $\{X_n(\omega)\}$ is increasing and, by virtue of a result of

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Chebyshev (see [1–4]),

$$X_n(\omega) = \max_{\pi_{n-1}} \frac{\displaystyle\int_{-\infty}^{\infty} x\pi_{n-1}^2(x)\omega(x)\mathrm{d}x}{\displaystyle\int_{-\infty}^{\infty} \pi_{n-1}^2(x)\omega(x)\mathrm{d}x}, \qquad n = 1, 2, \cdots, \tag{1.1}$$

the maximum being attained for $\pi_{n-1}(x) = \dfrac{p_n(\omega, x)}{x - X_n(\omega)}$. Here and in what follows π_k denotes an arbitrary real polynomial $\neq 0$ of degree $\leq k$. For even weight, (1.1) can be extended to

$$X_n^2(\omega) = \max_{\pi_{n-2}} \frac{\displaystyle\int_{-\infty}^{\infty} x^2\pi_{n-2}^2(x)\omega(x)\mathrm{d}x}{\displaystyle\int_{-\infty}^{\infty} \pi_{n-2}^2(x)\omega(x)\mathrm{d}x}, \qquad n = 1, 2, \cdots \tag{1.2}$$

The essence of relations (1.1) and (1.2) lies in the fact that Gauss quadrature formula always exists in the case of one dimension and all the weights are positive.

Let us turn to the case of higher dimension. Let Π^d be the space of all polynomials in d variables, and Π_n^d the subspace of polynomials of total degree $\leq n$. Let $\mathscr{L}$ be a positive definite linear functional acting on Π^d. According to [5], a cubature formula of degree $2n-1$ with Π_{n-1}^d nodes is called a Gaussian cubature formula and if $\{P_\alpha^n\}_{|\alpha|=n}$ has Π_{n-1}^d common zeros, then $\mathscr{L}$ is called Gaussian, where $\{P_\alpha^n\}_{|\alpha|=n}$ is a sequence of orthonormal polynomials with respect to $\mathscr{L}$. If $\mathscr{L}$ is Gaussian, then the following result holds.

Theorem 1.1[6] *Let $l(x_1, \cdots, x_d)$ be an arbitrary linear function with respect to $\{x_i\}_{i=1}^d$. Then*

$$\max\{l(X_i^n),\ X_i^n \in H_n\} = \max_{\pi_{n-1}} \frac{\mathscr{L}(l(x)\pi_{n-1}^2(x))}{\mathscr{L}(\pi_{n-1}^2(x))}, \qquad n = 1, 2, \cdots, \tag{1.3}$$

where $H_n = \{X_i^n\}_{i=1}^N$ are the mutually different common zeros of $\{P_\alpha^n\}_{|\alpha|=n}$.

However, the Gaussian cubature formulae in several variables rarely exist although they indeed exist in some special cases. And it was shown that there does not have Gaussian cubature formulae for the standard regions. For the example of Gaussian cubature formulae, we can refer to Dunkl and Xu[5].

In this article, we present another generalization of Chebyshev's maximum principle which does not need $\mathscr{L}$ to be Gaussian. Our results can be applied to the classical integration of several variables, such as the integration over the sphere, the triangle, the square and so on.

This article is organized as follows. In Section 2, we present some maximum formulae for the integration over the disk and a further extension is given in Section 3.

2 Chebyshev's Maximum Principle over the Unit Disk

It is well known that there does not have Gaussian cubature formula with respect to the integration over the disk, but Bojanov and Petrova[7–8] presented a nonstandard Gaussian quadrature. They studied quadrature formulae for the unit disk $D := \{(x, y) \mid x^2 + y^2 \leq 1\}$, which are based on integrals over n chords. Their main result can be written as