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Planar-busting Curves on the Boundary of a Handlebody

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Abstract: Let H_n be an orientable handlebody of genus n. It has been proved that for n not less than 2, there exists an annulus-busting curve in ∂H_n. In the present paper, we prove that for n not less than 2, there exists an essential simple closed curve C in ∂H_n which intersects each essential planar surface in H_n non-emptily. Furthermore, we show that for n not less than 3, a pants-busting curve must also be an annulus-busting curve.

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1 Introduction

Let H_n be an orientable handlebody of genus n. A planar surface in H_n is a 2-sphere with some holes. The relation between planar surfaces and thin positions is considered in [1–2].

Rubinstein and Scharlemann[3] considered the maximal essential annuli in H_2. Lei and Tang[4] detected the maximal essential annuli in H_n ($n \geq 2$). It is a result in [5] that for each $n \geq 2$, there exists an essential simple closed curve C on the boundary of H_n such that C intersects every essential annulus in H_n non-emptily. This curve C is called an annulus-busting curve.

In the present paper, we show the existence of planar-busting curves. Namely, for each $n \geq 2$, there exists an essential simple closed curve C on the boundary of H_n such that

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C intersects every essential planar surface in H_n non-emptily. Furthermore, we show that under some conditions, a pants-busting curve is also an annulus-busting curve.

In Section 2, we show some useful propositions, and we use them to prove the main results in Sections 3 and 4.

All the manifolds considered in the paper are assumed to be compact, orientable and connected. The definitions and terminologies not defined here are standard; see, for example, [6–7].

2 Preliminaries

Let H_n be an orientable handlebody of genus n. A connected properly embedded surface P in H_n is essential if P is incompressible and is not boundary parallel in H_n.

Definition 2.1 *Let H_n be a handlebody of genus n. An essential simple closed curve C in ∂H_n which intersects each essential pair of pants (annulus) in H_n non-emptily is called a pants-busting (annulus-busting) curve. An essential simple closed curve C in ∂H_n which intersects each essential planar surface in H_n non-emptily is called a planar-busting curve.*

From a result of Schultens[8], it is easy to see the following proposition.

Proposition 2.1 *Let H_n be a handlebody with genus $n \geq 2$, and P be an essential planar surface in H_n. Then the manifold obtained from cutting H_n open along P may be a handlebody or consist of two handlebodies.*

Let S be a closed orientable surface, α_0, α_1, $\cdots$, α_n be a sequence of essential simple closed curves in S such that for each $1 \leq i \leq n$, α_{i-1} and α_i can be isotoped to be disjoint. Then we say that the sequence is a path of length n.

The distance $d(\alpha, \ \beta)$ between a pair α, β of essential simple closed curves in S is the smallest integer n such that there is a path from α to β of length n. Let $M = V_1 \bigcup_S V_2$ be a Heegaard splitting of a 3-manifold M.

Denote by $d(S)$ the distance of the Heegaard splitting which is defined as

$$d(S) = \min\{d(C_1, \ C_2) \mid C_i \text{ bounds an essential disk in } V_i, \ i = 1, 2\}.$$

It is clear that $V_1 \bigcup_S V_2$ is reducible if and only if $d(S) = 0$, and $V_1 \bigcup_S V_2$ is weakly reducible if and only if $d(S) \leq 1$.

The following theorem of Hempel[9] is important for proving our main theorem.

Theorem 2.1[9] *For positive integers $m, n \geq 2$, there exists a Heegaard splitting $V_1 \bigcup_S V_2$ of genus n for a closed orientable 3-manifold M with the distance $d(S) > m$.*

For convenience, we give the following definition and we will use it later.

Definition 2.2 *Let H_n $(n \geq 2)$ be a handlebody of genus n, A be an essential annulus in H_n, and $\partial A = a \bigcup b$. Let α be a simple arc in ∂H_n with $\alpha \bigcap A = \alpha \bigcap a = \partial\alpha$ (or $\alpha \bigcap A =$*