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On Some Varieties of Soluble Lie Algebras*

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Abstract: In this paper, we study a class of soluble Lie algebras with variety relations that the commutator of m and n is zero. The aim of the paper is to consider the relationship between the Lie algebra L with the variety relations and the Lie algebra L which satisfies the permutation variety relations for the permutation φ of $\{3, \cdots, k\}$.

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1 Introduction

There are many parallel results between groups and Lie algebras. We can translate some results from groups to Lie algebras. For example, Macdonald[1] discussed some varieties of groups, particularly, some varieties associated with nilpotent groups in 1961, and then Suthathip[2] showed the similar varieties for nilpotent Lie algebras. In this paper, we extend similar varieties in [3] to soluble Lie algebras.

Let L be a Lie algebra, and $x_1, x_2, \cdots, x_n \in L$. The commutator $[x_1, x_2, \cdots, x_n]$ in L is defined by

$$[x_1, x_2] = [x_1, x_2]$$

and

$$[x_1, x_2, \cdots, x_{n-1}, x_n] = [[x_1, x_2, \cdots, x_{n-1}], x_n], \qquad n \geq 2. \tag{1.1}$$

Moreover, we define

$$[x_1, x_2, \cdots, x_m;\ y_1, y_2, \cdots, y_n] = [[x_1, x_2, \cdots, x_m],\ [y_1, y_2, \cdots, y_n]]$$

for any integers m and n. We say that the Lie algebra L is variety $[m, n] = 0$ if it satisfies

$$[[x_1, x_2, \cdots, x_m],\ [y_1, y_2, \cdots, y_n]] = 0, \qquad x_i, y_j \in L.$$

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If a Lie algebra L satisfies $[x_1, x_2, \cdots, x_k] = [x_1, x_2, x_{\varphi(3)}, \cdots, x_{\varphi(k)}]$, where φ is a permutation of $\{3, \cdots, k\}$, then we call that L satisfies $C(k, \varphi)$. If L satisfies $C(k, \varphi)$ for all permutations φ of $\{3, \cdots, k\}$, then we call that L satisfies $C(k)$.

The main result of this paper is that L satisfies $C(n+2)$ $(n \geq 2)$ if and only if L satisfies the law $[n-k, 2+k] = 0$ for all $k = 0, 1, \cdots, n-2$. Then it is easy to see that [3,2]=0 is equivalent to $C(5)$. Furthermore, $[n, 2] = 0$ $(n \geq 3)$ implies $C(2n-1)$. However, the law $[m, n] = 0$ does not imply any nontrivial law $C(k, \varphi)$ for $m, n \geq 3$.

2 The Lie Algebra with Varieties $[m, n] = 0$

Now we want to introduce some properties of the Lie algebra with variety $[m, n] = 0$. Denote by (x) a subalgebra generated by x.

Definition 2.1 *Let L be a Lie algebra. We define the sequence $\{L^n\}_{n\geq 1}$ by*

$$L^1 = L, \qquad L^{n+1} = [L, L^n], \qquad n \geq 1.$$

If $L^{m+1} = 0$, $L^m \neq 0$ for some m, then we say that L has nilpotent class precisely m.

Lemma 2.1[4] *Let A be an associative algebra. Then the following identities hold*:

(1) $(\mathrm{ad}c)^m(a) = \sum\limits_{0\leq j\leq m} (-1)^{m-j} \begin{pmatrix} m \\ j \end{pmatrix} c^j a c^{m-j}$ for all $a, c \in A$;

(2) $[ab, c] = [a, c]b + a[b, c]$ for all $a, b, c \in A$.

Lemma 2.2[3] *If L satisfies $[n, m] = 0$, then $[n+p, m+q] = 0$ for any nonnegative numbers p and q.*

Lemma 2.3[5] *If L satisfies $C(k, \varphi_1)$ and $C(k, \varphi_2)$, then L satisfies $C(k, \varphi)$ for any φ in the group generated by φ_1 and φ_2.*

Lemma 2.4[5] *If L satisfies $C(k)$, then L satisfies $C(m)$ for all $m \geq k$.*

Lemma 2.5 *Let L be a Lie algebra. Then $[a, [x, y]] = 0$ if and only if $[a, x, y] = [a, y, x]$ for any $a, x, y \in L$.*

Proof. It is easily checked by Jacobian identity.

Lemma 2.6 *Let L be a Lie algebra with variety $[n, 2] = 0$ $(n \geq 2)$. If $L/Z(L)$ satisfies $C(n+1)$, then L satisfies $C(n+2)$.*

Proof. By Lemma 2.5, we know that L satisfies $C(n+2, \varphi_1)$ for $\varphi_1 = (n+1, n+2)$. Since $L/Z(L)$ satisfies $C(n+1)$, in particular, it satisfies $C(n+1, \varphi_2)$ for $\varphi_2 = (3, 4, \cdots, n+1)$. Thus, for any $x_1, x_2, \cdots, x_{n+1} \in L$, we have

$$[x_1, x_2, x_3, \cdots, x_{n+1}] - [x_1, x_2, x_{\varphi_2(3)}, \cdots, x_{\varphi_2(n+1)}] \in Z(L),$$

and also

$$[x_1, x_2, \cdots, x_{n+1}, x_{n+2}] = [x_1, x_2, x_{\varphi_2(3)}, \cdots, x_{\varphi_2(n+1)}, x_{\varphi_2(n+2)}]$$

for any $x_{n+2} \in L$. That is, L satisfies $C(n+2, \varphi_2)$. Since $S_n = \langle \varphi_1, \varphi_2 \rangle$, by Lemma 2.3, we know that L satisfies $C(n+2)$.