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On Generalized PST-groups*

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Abstract: A finite group G is called a generalized PST-group if every subgroup contained in $F(G)$ permutes all Sylow subgroups of G, where $F(G)$ is the Fitting subgroup of G. The class of generalized PST-groups is not subgroup and quotient group closed, and it properly contains the class of PST-groups. In this paper, the structure of generalized PST-groups is first investigated. Then, with its help, groups whose every subgroup (or every quotient group) is a generalized PST-group are determined, and it is shown that such groups are precisely PST-groups. As applications, T-groups and PT-groups are characterized.

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1 Introduction

Let G be a finite group. A subgroup H of G is said to be s-permutable in G, if H permutes all Sylow subgroups of G. It is well-known that an s-permutable subgroup is subnormal (see [1]). In this paper, we call G a generalized PST-group if every subgroup of G contained in $F(G)$ is s-permutable in G, where $F(G)$ is the Fitting subgroup of G. Agrawal[2] and many other authors have studied the so-called PST-group, i.e., group in which subnormal subgroups are s-permutable (see, e.g., [3]–[5]). Since subgroups of G contained in $F(G)$ are subnormal in G, it is clear that PST-groups are certainly generalized PST-groups. But the converse is not true. A counterexample is as follows:

Example 1.1 Suppose that $A = \langle a \rangle$ is a cyclic group of order 7 and

$$S_3 = \langle b, c \mid b^3 = c^2 = 1,\ b^c = b^{-1} \rangle$$

is the symmetric group on three letters. Let

$$H = A \times S_3.$$

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Clearly, the mapping

$$a \mapsto a^2, \quad b \mapsto b, \quad c \mapsto c$$

determines an automorphism α of H of order 3. Let G be the semidirect product of H and $\langle\alpha\rangle$. Then

(i) $F(G) = \langle a\rangle \times \langle b\rangle$, and every subgroup of $F(G)$ is normal in G. Hence G is a generalized PST-group.

(ii) Let

$$K = S_3 \times \langle\alpha\rangle.$$

Clearly,

$$F(K) = \langle b\rangle \times \langle\alpha\rangle.$$

Since $\langle b\alpha\rangle$ cannot permute the Sylow 2-subgroup $\langle c\rangle$ of K, K is not a generalized PST-group.

(iii) As $G/A \cong K$, G/A is not a generalized PST-group.

(iv) G is not a PST-group because any quotient group of a PST-group is still a PST-group.

From the above example we see that the class of generalized PST-groups is not subgroup and quotient group closed, and it properly contains the class of PST-groups. So it is meaningful to investigate the generalized PST-groups. Especially, we are interested in determining the groups in which every subgroup (or every quotient group) is still a generalized PST-group. The result we obtain is surprising: those groups are precisely PST-groups. In our investigation, power automorphisms play an important role. A power automorphism of a group G is an automorphism that leaves every subgroup of G invariant. Such an automorphism maps each element to one of its powers. The main result is the following:

Theorem 1.1 *Let G be a finite group. Then the following conditions are equivalent*:

(i) *Every subgroup of G is a PST-group*;

(ii) *Every subgroup of G is a generalized PST-group*;

(iii) *The nilpotent residue L of G, i.e., the smallest term of the lower central series of G, is an abelian Hall subgroup of G of odd order, and every element of G induces a power automorphism in L*;

(iv) *G is a solvable PST-group*;

(v) *G is solvable and every quotient group of G is a generalized PST-group.*

A direct consequence of Theorem 1.1 is the following corollary:

Corollary 1.1 *A non-solvable PST-group has a subgroup which is not a PST-group.*

Let G be a finite group. G is called a T-group if every subnormal subgroup of G is normal in G, and G is called a PT-group if every subnormal subgroup of G permutes all subgroups of G. Clearly, T-groups and PT-groups are both PST-groups. By Theorem 1.1, Lemma 13.4.5 of [6] and Lemma 1 of [7], we can easily obtain the following results: