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Existence of Anti-periodic Solution for Differential Equations*

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Abstract: In this paper, we discuss the anti-periodic boundary value problem for a class of first order differential equations. By using homotopy method, we obtain the conditions for the existence of anti-periodic solution for the equation under consideration. This result can be extended to higher order differential equations.

Key words: continuation theorem, anti-periodic solution, homotopy method

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1 Introduction

It is well known that the existence and uniqueness of solutions to boundary value problems for differential equation has always been an important issue. Some theories and methods have been introduced in [1], such as the upper and lower solutions method, monotone iterative technique, fixed point theorem, variational method and critical point theory. Anti-periodic boundary problems have been studied widely in the last ten years. For example, for first order ordinary differential equation, Chen[2] builds the Massera type criterion. Some existence condition for first-order ordinary differential equation can be found in [3] and [4]. On the base of Leray-Shauder type argument, the existence and uniqueness results are obtained for higher-order ordinary differential equation in [5] and [6]. There are similar results for partial differential equation and abstract differential equation, see [7] and [8] for detail. Abdurrahman[9] and Ahn[10] proved the existence of the anti-periodic boundary value problems for impulsive differenitail equations. Some recent related works can be found in [11]–[13].

The main purpose of this paper is to consider the existence of anti-periodic solutions to

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the nonlinear first order differential equation

$$x' = f(t, x), \qquad t \in [0,\ T], \tag{1.1}$$

where $f : \mathbf{R}^1 \times \mathbf{R}^n \to \mathbf{R}^n$ is continuous and satisfies $f(t+T, x) = -f(t, -x)$, and T is a positive constant.

For some $T > 0$, the equation (1.1) is said to be dissipative if there exist $B > 0$ and $A(r), K(r) > 0$, such that $|x(0)| \le r$ implies that $|x(t)| \le A(r)$, $t \ge 0$, $|x(t)| \le B$, $t \ge K(r)$.

A solution $x(t)$ of (1.1) is said to be T-antiperiodic if $x(t+T) = -x(t),\ t \in \mathbf{R}$.

The outline of this paper is as follows. In Section 2 we state and prove our main results by homotopy method. A similar result for higher order differential equations are given in Section 3.

2 Main Result

Consider the equation

$$x' = f(t, x), \tag{2.1}$$

where $f : \mathbf{R}^1 \times \mathbf{R}^n \to \mathbf{R}^n$ is continuous with $f(t+T, x) = -f(t, -x)$.

Theorem 2.1 *Let $\Omega \subset \mathbf{R}^n$ be a bounded open set with $0 \in \Omega$. Consider the auxiliary equation*

$$x' = \lambda f(t, x), \qquad \lambda \in [0, 1]. \tag{2.2}$$

If for $\lambda \in (0, 1]$, every possible T-anti-periodic $x(t)$ of the equation (2.2) satisfies $x(t) \notin \partial\Omega$ for all $t \in [0, T]$, then (2.1) has a T-anti-periodic solution $x(t) \in \Omega$, $t \in [0, T]$.

Proof. Firstly, we assume that f is C^2 with respect to x, and then the solution of equation (2.1) with initial value is unique. Denote by $x(t, x_0, \lambda)$ the solution of equation of (2.2) with the initial value condition $x(0) = x_0$.

If $x(t)$ is a T-anti-periodic solution of (2.2), then $x(0) + x(T) = 0$, i.e.,

$$x(0) + x(0) + \lambda \int_0^T f(s, x(s))\mathrm{d}s = 0,$$

which is equivalent to

$$x(0) = -\frac{\lambda}{2} \int_0^T f(s, x(s))\mathrm{d}s.$$

Consider the homotopy equation

$$H(x_0, \bar{x}_0, \lambda) = x_0 + \frac{\lambda}{2} \int_0^T f(s, x(s, x_0, \lambda))\mathrm{d}s - (1-\lambda)\bar{x}_0,$$

where $\bar{x}_0 \in S_r(0) = \{P \in \mathbf{R}^n :| P |< r\} \subset \Omega$. By the assumptions of the theorem, it is not difficult to see that there exists a sufficiently small r, such that

$$0 \notin H(\partial\Omega, \bar{x}_0, [0, 1]). \tag{2.3}$$

Note that

$$\operatorname{rank}(H_{x_0}, H_{\bar{x}_0}, H_\lambda) = \operatorname{rank}(H_{x_0}, -(1-\lambda)I, H_\lambda) = n, \qquad \lambda \in [0, 1).$$