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Norm Estimates for the Inverses of Strictly Diagonally Dominant M-Matrices and Linear Complementarity Problems

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Abstract. A partition reduction method is used to obtain two new upper bounds for the inverses of strictly diagonally dominant M-matrices. The estimates are expressed via the determinants of third order matrices. Numerical experiments with various random matrices show that they are stable and better than the estimates presented in literature. We use these upper bounds in order to improve known error estimates for linear complementarity problems with B-matrices.

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Key words: Strictly diagonally dominant matrix, M-matrix, linear complementarity problem, inverse, infinity norm bound.

1. Introduction

A matrix $A=(a_{ij})\in\mathbb{C}^{n\times n}$ is called diagonally dominant (DD), if

$$|a_{ii}|\geq\sum_{i\neq j}|a_{ij}|\equiv r_i(A),\quad i=1,2,\dots,n, \tag{1.1}$$

and strictly diagonally dominant (SDD) if the inequalities in (1.1) all hold strictly. If A is an SDD matrix, the diagonal dominance degree [39] of A with respect to i-th row is defined as $|a_{ii}|-r_i(A)$. SDD matrices play an important role in information theory, system theory, modern economics, network, algorithm and programming — cf. [3,10,12,26,30,44]. Various properties of SDD matrices such as nonsingularity [27,28], the infinity norm estimates of their inverses [35,47,48], Schur complement problem [32,38], and the errors of linear complementarity problems [20,42] have been already studied. The SDD M-matrices — i.e.

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matrices with positive diagonal elements and nonpositive all others, are of special interest since they are related to the stiffness matrices arising in finite difference and finite element methods for partial differential equations [8]. Besides, such matrices appear in economics and networks.

Although the norm $\|A\|_\infty$ can be easily determined with a high accuracy, for large A the computation of A^{-1} and $\|A^{-1}\|_\infty$ requires substantial efforts. Therefore, the aim of this work is to present sharp estimates of the norm $\|A^{-1}\|_\infty$. This is important for applications. Thus the condition number $\kappa_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty$ is a measure of the stability and sensitivity of A and is used in error analysis, in the study of perturbations of eigenvalues and in iterative algorithms for linear systems [17,46]. Besides, the corresponding norm estimates appear when studying the convergence of matrix splitting and matrix multi-splitting iterative methods for large sparse systems of linear equations [48] and in establishing lower bounds for singular values of matrices [11,40,47]. Thus Chen and Xiang [5] presented error estimates for linear complementarity problems based on the infinity norm estimates of the related matrices. The infinity norms estimate of inverse matrices are also used in measuring the distance to instability [31,42].

Note that for strictly diagonally dominant matrices $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, Varah [47] obtained the estimate

$$\|A^{-1}\|_\infty \le \frac{1}{\min\limits_i \{|a_{ii}| - r_i(A)\}}. \tag{1.2}$$

Later on, Cheng and Huang [7] pointed out that if the minimum of the diagonal dominance degree of A is small, the right-hand side of the inequality (1.2) can be very large. However, for strictly diagonally dominant M-matrices they provide another estimate — viz.

$$\|A^{-1}\|_\infty \le \frac{1}{a_{11}(1-u_1 l_1)} + \sum_{i=2}^{n} \left[\frac{1}{a_{ii}(1-u_i l_i)} \prod_{j=1}^{i-1} \left(1 + \frac{u_j}{1-u_j l_j} \right) \right], \tag{1.3}$$

where

$$u_i = \frac{\sum\limits_{j=i+1}^{n} |a_{ij}|}{|a_{ii}|}, \quad l_i = \max_{i\le s\le n} \left\{ \frac{\sum\limits_{i\le t\ne s\le n} |a_{st}|}{|a_{ss}|} \right\}.$$

It was also proved that the estimate (1.3) is sharper than the one in [45]. Wang [50] improved (1.3) to

$$\|A^{-1}\|_\infty < \frac{1}{a_{11}(1-u_1 l_1)} + \sum_{i=2}^{n} \left[\frac{1}{a_{ii}(1-u_i l_i)} \prod_{j=1}^{i-1} \left(\frac{1}{1-u_j l_j} \right) \right], \tag{1.4}$$

and then obtained a sharper estimate — viz.

$$\|A^{-1}\|_\infty < \left(a_{11} - \sum_{k=2}^{n} |a_{1k}| \omega_{k1} \right)^{-1} + \sum_{i=2}^{n} \left(a_{ii} - \sum_{k=i+1}^{n} |a_{ik}| \omega_{ki} \right)^{-1} \prod_{j=1}^{i-1} \left(\frac{1}{1-u_j l_j} \right), \tag{1.5}$$