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On Second-order Sufficient Conditions in Constrained Nonsmooth Optimization*

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Abstract: In this paper, we establish a second-order sufficient condition for constrained optimization problems of a class of so called ℓ-stable functions in terms of the first-order and the second-order Dini type directional derivatives. The result extends the corresponding result of [D. Bednařík and K. Pastor, Math. Program. Ser. A, 113(2008), 283–298] to constrained optimization problems.

Key words: second-order optimality condition, ℓ-stable function, Dini directional derivative, isolated minimizer

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1 Introduction and Preliminaries

Second-order optimality conditions play a crucial role in optimization theory. For example, they are very useful for the study of sensitivity analysis of optimal solutions and convergence analysis of optimal algorithms. Various second-order directional derivatives have been introduced and studied for developing second-order optimality conditions (see [1]–[14] and reference therein). Especially, Ginchev[10] gave second-order sufficient and necessary optimality conditions for unconstrained optimality problems in terms of Hadamard type derivatives and Huang[12] presented separate sufficient and necessary optimality conditions for a constrained optimization problem in terms of Hadamard type derivatives. As shown in [13] that, even for the class of twice differentiable functions (C^2), the classical second-order derivative does not coincide the second-order Hadamard derivative introduced in [10], while there is a coincidence for the Dini derivative. Therefore it is nature to consider a

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class of functions for which the second-order optimality conditions can be formulated with the Dini derivatives instead of the Hadamard derivatives. Ginchev *et al.*[13] showed that the Hadamard derivatives in second-order optimality condition given in [10] can be replaced by the Dini type directional derivatives for the class of $C^{1,1}$ functions. The class of $C^{1,1}$ functions (differentiable functions whose derivatives are locally Lipschitz) was first brought to attention by Hiriart-Urruty in 1977 (see [2]). Recently, Bednařík and Pastor[14] have attempted to weaken the $C^{1,1}$ assumption and studied so called ℓ-stable function in order to give second-order sufficient optimality conditions for unconstrained optimization problem with the Dini derivatives.

The aim of this paper is to extend the result obtained in [14] to the case of constrained optimization problems, that is, to establish second-order sufficient optimality conditions for constrained optimization problem with ℓ-stable functions in terms of the Dini derivatives.

In the following, we denote by $\|\cdot\|$, $S_{\mathbf{R}^n} = \{x \in \mathbf{R}^n; \|x\| = 1\}$, and $\langle\cdot, \cdot\rangle$, the norm, the unit sphere, and the scalar product of $\mathbf{R}^n$, respectively. We also denote by $B(x, \delta)$ the open ball centered at x with radius δ, and always identify the space $\mathbf{R}^n$ with its dual space.

For a function $f : \mathbf{R}^n \to \mathbf{R}$, the first-order lower Dini directional derivative of f at $x \in \mathbf{R}^n$ in the direction $u \in \mathbf{R}^n$, and the first-order lower Hadamard directional derivative of f at $x \in \mathbf{R}^n$ in the direction $u \in \mathbf{R}^n$ are defined by

$$f'_-(x, u) = \liminf_{t\downarrow 0} \frac{f(x + tu) - f(x)}{t},$$

and

$$f^{\downarrow}_-(x, u) = \liminf_{(t,v)\to(+0,u)} \frac{f(x + tv) - f(x)}{t}$$

respectively. If the function f is locally lipschitz, then

$$f'_-(x, u) = f^{\downarrow}_-(x, u).$$

The second-order lower Dini directional derivative of $f : \mathbf{R}^n \to \mathbf{R}$ at $x \in \mathbf{R}^n$ in the direction $u \in \mathbf{R}^n$ is given as

$$f''_-(x, u) = \liminf_{t\downarrow 0} \frac{f(x + tu) - f(x) - tf'_-(x, u)}{\frac{t^2}{2}},$$

and the second-order lower Hadamard directional derivative of $f : \mathbf{R}^n \to \mathbf{R}$ at $x \in \mathbf{R}^n$ in the direction $u \in \mathbf{R}^n$ is given as

$$f^{\downarrow\downarrow}_-(x, u) = \liminf_{(t,v)\to(+0,u)} \frac{f(x + tv) - f(x) - tf^{\downarrow}_-(x, u)}{\frac{t^2}{2}}.$$

Consider the following constrained optimization problem:

$$\text{(P)} \qquad \min_{x\in S} f(x),$$

where S is a closed subset in $\mathbf{R}^n$, and $f : \mathbf{R}^n \to \mathbf{R}$ is a real-valued function.

Recall that a point $x \in S \subset \mathbf{R}^n$ is said to be a local minimum of f over S if there exists $\delta > 0$ such that

$$f(y) \geq f(x), \qquad \forall y \in S \cap B(x, \delta). \tag{1.1}$$