

LG/CY Correspondence Between tt^* Geometries

Huijun Fan*, Tian Lan and Zongrui Yang

School of Mathematical Sciences, Peking University, Beijing, China.

Received 1 December 2020; Accepted 18 February 2021

Dedicated to Prof. Armen Glebovich Sergeev for his 70's birthday.

Abstract. The concept of tt^* geometric structure was introduced by physicists (see [4, 10] and references therein), and then studied firstly in mathematics by C. Hertling [28]. It is believed that the tt^* geometric structure contains the whole genus 0 information of a two dimensional topological field theory. In this paper, we propose the LG/CY correspondence conjecture for tt^* geometry and obtain the following result. Let $f \in \mathbb{C}[z_0, \dots, z_{n+1}]$ be a nondegenerate homogeneous polynomial of degree $n+2$, then it defines a Calabi-Yau model represented by a Calabi-Yau hypersurface X_f in $\mathbb{C}P^{n+1}$ or a Landau-Ginzburg model represented by a hypersurface singularity (\mathbb{C}^{n+2}, f) , both can be written as a tt^* structure. We proved that there exists a tt^* substructure on Landau-Ginzburg side, which should correspond to the tt^* structure from variation of Hodge structures in Calabi-Yau side. We build the isomorphism of almost all structures in tt^* geometries between these two models except the isomorphism between real structures.

AMS subject classifications: 14D05, 14D07, 32W99, 58K99

Key words: tt^* geometry, Landau-Ginzburg/Calabi-Yau correspondence, variation of Hodge structures.

1 Introduction

In the early of 1980's, two physicists, B. Greene and R. Plesser [23] found a mysterious duality of Hodge numbers between two T-dual Calabi-Yau 3-folds M and \check{M}

*Corresponding author. *Email addresses:* fanhj@math.pku.edu.cn (H. Fan), 1701110008@pku.edu.cn (T. Lan), 1600010619@pku.edu.cn (Z. Yang)

(called mirror pair): $h^{3-p,q}(M) = h^{p,q}(\check{M})$. In particular, there is $h^{1,1}(M) = h^{2,1}(\check{M})$. This observation opened the gate to the vast field of mirror symmetry. The equality reflects the equivalence between two quantized theories. One is realized by quantum invariants of symplectic geometry (called A theory or A-model) and the other is realized by quantum invariants of complex geometry in the dual Calabi-Yau manifold (called B theory or B-model). The quantum theory of symplectic geometry is just the Gromov-Witten theory, whose all genus invariants were already defined. The higher genus quantum invariants of the complex geometry have not been built rigorously in mathematics despite of much computation in physical side (see [4, 30] and consequent work). The genus 0 invariant of the complex geometry of a Calabi-Yau manifold was understood clearly, which is related to the deformation theory of the complex structure, involving the "classical" geometrical structures such as variation of Hodge structure, Gauss-Manin connection, period integration and etc. The proof of genus 0 mirror symmetry conjecture is the main theorem of the long monograph [29] which elucidates the mirror symmetry phenomenon from the mathematical and physical sides. Since then, there are much work concerning the proof of mirror symmetry conjecture between Gromov-Witten theory and the physical BCOV's theory [52]. Though there are some efforts [15] to build the higher genus invariant based on the genus 0 theory, the complete and rigorous mathematical theory is still unknown.

The A theory of a Calabi-Yau (CY) model is the Gromov-Witten theory that corresponds to the Calabi-Yau nonlinear sigma model in physics. The minimizers of the Lagrangian is the holomorphic maps from the genus g Riemann surfaces to the Calabi-Yau manifold. There is another much related model, called Landau-Ginzburg (LG) model, which consists of a noncompact Kahler manifold with a holomorphic function over it, called superpotential. The A theory of LG model was constructed by the first author, Jarvis and Ruan based on Witten's r -spin theory, called FJRW theory (see [19] and consequent work). Now people realize that the mirror symmetry is not only the dual equivalence between A theory and B theory of Calabi-Yau mirror pair, but forms a sophisticated duality relation between the A and B theories of CY model and LG model. This global dual relation can be briefly described by the following diagram:

$$\begin{array}{ccc}
 \text{LG A theory} & \xrightarrow{\text{mirror}} & \text{LG B theory} \\
 \downarrow \text{LG/CY corresp.} & & \downarrow \text{LG/CY corresp.} \\
 \text{CY A theory} & \xrightarrow{\text{mirror}} & \text{CY B theory}
 \end{array} \tag{1.1}$$