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Anisotropic Elliptic Nonlinear Obstacle Problem with Weighted Variable Exponent

Adil Abbassi*, Chakir Allalou and Abderrazak Kassidi

LMACS Laboratory, Mathematics Department, Faculty of Sciences and Techniques, Sultan Moulay Slimane University Beni-Mellal, BP: 523, Morocco.

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Abstract. In this paper, we are concerned with a show the existence of a entropy solution to the obstacle problem associated with the equation of the type :

$$\begin{cases} Au+g(x,u,\nabla u)=f & \text{in} \quad \Omega, \\ u=0 & \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N\geq 2$, A is an operator of Leray-Lions type acting from $W_0^{1,\overrightarrow{p}(.)}(\Omega,\ \overrightarrow{w}(.))$ into its dual $W_0^{-1,\overrightarrow{p}'(.)}(\Omega,\ \overrightarrow{w}^*(.))$ and L^1-deta. The nonlinear term g: $\Omega\times\mathbb{R}\times\mathbb{R}^N\longrightarrow\mathbb{R}$ satisfying only some growth condition.

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Key words: Entropy solutions, Anisotropic elliptic equations, weighted anisotropic variable exponent Sobolev space.

1 Introduction

Consider Ω be a bounded open subset of $\mathbb{R}^N(N\geq 2)$ and $p_i(.)\in C_+(\overline{\Omega})$ for $i=0,1,\cdots,N$, with for all x in Ω,

$$p_0(x)\geq \max\{p_i(x),\ i=1,\ \cdots,\ N\}. \tag{1.1}$$

Let $W^{1,\overrightarrow{p}(.)}(\Omega,\overrightarrow{w}(.))$ be weighted anisotropic variable exponent Sobolev space associated to the vector $\overrightarrow{p}(.)$, with $\overrightarrow{p}(.)=\{p_0(\cdot),\cdots,p_N(.)\}$, where $p_0(x),p_1(x),\cdots,p_N(x)$ be $N+1$ variable exponents and $\overrightarrow{w}(.)$ denoting a vector of measurable positive functions, i.e., $\overrightarrow{w}(.)=\{w_1(\cdot),\cdots,w_N(.)\}$, with w_i are weight measurable functions for all $i=1,\cdots,N$.

*Corresponding author. *Email addresses:* abbassi91@yahoo.fr (A. Abbassi), chakir.allalou@yahoo.fr (C. Allalou), abderrazakassidi@gmail.com (A. Kassidi)

Our aim is to prove the existence of solutions with respect to perturbations in the growth exponent p of the following problems:

$$\begin{cases} -\sum_{i=1}^{N} D^i a_i(x,u,\nabla u) + g(x,u,\nabla u) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{$\mathcal{P}$}$$

in the convex class $K_\psi := \left\{ u \in W_0^{1,\overrightarrow{p}(x)}(\Omega, \overrightarrow{w}(x)), u \geq \psi \quad \text{a.e in } \Omega \right\}$, where ψ is a fixed obstacle function, such that

$$\psi^+ \in W_0^{1,\overrightarrow{p}(x)}(\Omega, \overrightarrow{w}(x)) \cap L^\infty(\Omega). \tag{1.2}$$

We assume that $a_i: \Omega \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}$ are Carathodory functions for $i = 1,2,\cdots,N$, (measurable with respect to x in Ω for every (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$ and continuous with respect to (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$ for almost every x in Ω) which satisfies the following conditions:

$$a_i(x, s, \xi)\xi_i \geq \alpha w_i \left|\xi_j\right|^{p_i(x)} \qquad \text{for} \quad i = 1,\cdots,N, \tag{1.3}$$

$$|a_i(x, s, \xi)| \leq \beta w_i^{\frac{1}{p_i(x)}} \left(M_i(x) + |s|^{p_i(x)-1} + w_i^{\frac{1}{p_i'(x)}} |\xi_i|^{p_i(x)-1} \right) \quad \text{for} \quad i = 1,\cdots,N, \tag{1.4}$$

for all $\xi = (\xi_1, \cdots, \xi_N)$ and $\xi' = (\xi_1', \cdots, \xi_N')$, we have

$$(a_i(x, s, \xi) - a_i(x, s, \xi'))(\xi_i - \xi_i') > 0 \quad \text{for} \quad \xi_i \neq \xi_i', \tag{1.5}$$

for a.e. $x \in \Omega$, and all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, where $M_i(.)$ is a nonnegative function lying in $L^{p_i'(.)}(\Omega)$ and $\alpha, \beta > 0$.

The nonlinear term $g(x, s, \xi)$ is a Caratheodory function which satisfies only the growth condition

$$|g(x, s, \xi)| \leq c(x) + b(|s|) \sum_{i=1}^{N} w_i |\xi_i|^{p_i(x)} \tag{1.6}$$

where $b: \mathbb{R} \mapsto \mathbb{R}^+$ is a continuous positive function that belongs to $L^1(\mathbb{R})$ and $c(x) \in L^1(\Omega)$.

In the particular case when $p_i = p$ for any $i \in \{1,...,N\}$, Yazough, Azroul and Redwane (see [16]) have proved the existence of entropy solutions to problem like $(\mathcal{P})$. Then, Azroul, Benboubker and Ouaro [6] have obtained the above results via penalization methods.

The study of $(\mathcal{P})$ is a new and interesting topic when the data is in L^1. One result on this topic can be found in [5,8,11], where the discussion was conducted in the framework of weighted anisotropic Sobolev space with variable exponent (we refer to [1,2,11] for more details), the notion of a entropy solution was introduced by Benilan et. al [7,9] and P.-L. Lions [14] in their study of the Boltzmann equation. We mention some works in the direction of the anisotropic space such as [4,8].