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On the h-almost Yamabe Soliton

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Abstract. We introduce the concept h-almost Yamabe soliton which extends naturally the almost Yamabe soliton by Barbosa-Ribeiro and obtain some rigidity results concerning h-almost Yamabe solitons. Some condition for a compact h-almost Yamabe soliton to be a gradient soliton is also obtained. Finally, we give some characterizations for a special class of gradient h-almost Yamabe solitons.

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1 Introduction

The Yamabe flow was introduced by Hamilton at the same time as the Ricci flow as an attempt to solve the Yamabe problem on manifolds of positive conformal Yamabe invariant. Formally, the Yamabe flow deforms a given manifold by evolving its metric according to

$$\frac{\partial}{\partial t}g(t)=-R(t)g(t), \tag{1.1}$$

where $R(t)$ denotes the scalar curvature of the metric $g(t)$. The Yamabe flow and the Ricci flow are equivalent in dimension $n=2$, but they are essentially different in higher dimensions [1].

A family of metrics $g(t)=\sigma(t)\psi_t^*g(0)$ solving (1.1), where $\sigma(t)$ is a positive smooth function and $\psi_t: M\to M$ is a one-parameter family of diffeomorphisms of M, is said to be a self-similar solution of the Yamabe flow. Yamabe solitons are self-similar solutions of the Yamabe flow. A Riemannian manifold (M^n,g) is a Yamabe soliton if it admits a vector field X such that

$$(R-\rho)g=\frac{1}{2}\mathcal{L}_Xg, \tag{1.2}$$

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where $\mathcal{L}_X$ denotes the Lie derivative in the direction of the vector field X and ρ is a real number. For $\rho=0$ the Yamabe soliton is steady, for $\rho<0$ is expanding, and for $\rho>0$ is shrinking. It has been known (see [2]) that a compact Yamabe soliton has constant scalar curvature, thus trivial. For more details on Yamabe soliton we refer the reader to [3–5].

Barbosa and Ribeiro introduced the almost Yamabe soliton in [6] as follows. A Riemannian manifold (M^n,g) is an almost Yamabe soliton if there exists a complete vector field X and a smooth soliton function ρ on (M^n,g) satisfying

$$(R-\rho)g=\frac{1}{2}\mathcal{L}_X g. \tag{1.3}$$

From the definition, if ρ is constant, almost Yamabe solitons are Yamabe solitons.

Recently, Gomes, Wang and Xia [7] have introduced the definition of the h-almost Ricci soliton. Such a soliton is a generalization of an almost Ricci soliton given by the authors in [8]. An h-almost Ricci soliton is a complete Riemannian manifold (M^n,g) with a vector field $X\in\mathfrak{X}(M)$, a soliton function $\lambda: M\to\mathbb{R}$ and a function $h: M\to\mathbb{R}$ which are smooth and satisfy the equation

$$Ric+\frac{h}{2}\mathcal{L}_X g=\lambda g. \tag{1.4}$$

In particular, the results contained in [7,9] indicates that h-almost Ricci solitons should reveal a reasonably broad generalization of the fruitful concept of the classical Ricci soliton, almost Ricci soliton and quasi Einstein manifolds [10–14]. For further details about h-almost Ricci soliton, see [7,9].

Therefore, it is very interesting to consider a generalization of the almost Yamabe solitons.

Definition 1.1. We say that a Riemannian manifold (M^n,g) is an h-almost Yamabe soliton if there exists a complete vector field X, a smooth soliton function $\rho: M\to\mathbb{R}$ and a signal function $h: M\to\mathbb{R}$ which satisfy the equation:

$$(R-\rho)g=\frac{h}{2}\mathcal{L}_X g, \tag{1.5}$$

where R denotes the scalar curvature of M^n. The function h is said to have a defined signal if either $h>0$ or $h<0$ on M.

In the following, we denote the h-almost Yamabe soliton satisfying (1.5) by (M^n,g,X,h,ρ). When ρ is constant the structure is said to be an h-Yamabe soliton. Moreover, if $X=\nabla u$, we call the equation

$$(R-\rho)g=\frac{h}{2}\mathcal{L}_{\nabla u} g, \tag{1.6}$$

a gradient h-almost Yamabe soliton, where $\nabla^2 u$ denotes the Hessian of u. An h-almost Yamabe soliton is said to be shrinking, steady or expanding if it admits a soliton field for which, respectively, $\rho>0$, $\rho=0$ or $\rho<0$. Otherwise, it will be called indefinite.