J. Math. Study
doi: 10.4208/jms.v54n4.21.06

Vol. **54**, No. 4, pp. 407-426
December 2021

A Fast Conservative Scheme for the Space Fractional Nonlinear Schrödinger Equation with Wave Operator

Mustafa Almushaira[1,2,3] and Fei Liu[1,2,*]

[1] *School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, China.*
[2] *Hubei Key Laboratory of Engineering Modeling and Scientific Computing, Huazhong University of Science and Technology, Wuhan 430074, China.*
[3] *Department of Mathematics, Faculty of Science, Sana'a University, Sana'a, Yemen.*

Received May 18, 2020; Accepted August 26, 2020;
Published online June, 25 2021.

Abstract. A new efficient compact difference scheme is proposed for solving a space fractional nonlinear Schrödinger equation with wave operator. The scheme is proved to conserve the total mass and total energy in a discrete sense. Using the energy method, the proposed scheme is proved to be unconditionally stable and its convergence order is shown to be of $\mathcal{O}(h^6+\tau^2)$ in the discrete L_2 norm with mesh size h and the time step τ. Moreover, a fast difference solver is developed to speed up the numerical computation of the scheme. Numerical experiments are given to support the theoretical analysis and to verify the efficiency, accuracy, and discrete conservation laws.

AMS subject classifications: 65M06, 65M12, 35Q41

Key words: Space-fractional nonlinear Schrödinger equations, fast difference solver, convergence, conservation laws.

1 Introduction

We consider the following space fractional nonlinear Schrödinger equation with wave operator (FNLSW)

$$u_{tt}+(-\Delta)^{\gamma/2}u+i\alpha u_t+\beta|u|^2u=0,\quad 1<\gamma\leq 2,\quad x\in\mathbb{R},\quad t\in(0,T],\tag{1.1}$$

with the initial conditions

$$u(x,0)=\phi_0(x),\quad u_t(x,0)=\phi_1(x),\quad x\in\mathbb{R},\tag{1.2}$$

*Corresponding author. *Email addresses:* `mstf1985@mail.hust.edu.cn` (M. Almushaira), `liufei@hust.edu.cn` (F. Liu)

and the periodic boundary conditions

$$u(x+L,t)=u(x,t), \qquad x\in\mathbb{R}, \quad t\in[0,T], \tag{1.3}$$

where $i=\sqrt{-1}$, α,β are both real constants, L is the period, $u(x,t)$ is an unknown complex function, $\phi_0(x)$ and $\phi_1(x)$ are known smooth functions. The fractional Laplacian operator in (1.1) is defined via the Fourier transform (see [4,7,18] and references therein) as

$$(-\Delta)^{\gamma/2}u(x)=\mathcal{F}^{-1}[|\xi|^{\gamma}\mathcal{F}[u](\xi)](x), \quad \forall x,\xi\in\mathbb{R}, \tag{1.4}$$

where $\mathcal{F}$ represents the Fourier transform acting on the spatial variable x , and $\mathcal{F}^{-1}$ denotes its inverse.

When $\gamma=2$, the FNLSW (1.1)-(1.3) can be regarded as a generalization of the classical nonlinear Schrödinger equation with wave operator (NLSW). NLSW is one of most important nonlinear Schrödinger-type equations which is widely used to describe many physical phenomena, such as nonlinear optics [3,21], plasma physics [19] and bimolecular dynamics [26].

The system (1.1)-(1.3) possesses at least two conserved quantities as shown in [20],

$$Q(t)=Q(0), \quad E(t)=E(0), \quad t\in[0,T], \tag{1.5}$$

where

$$\begin{aligned} Q(t)&=\frac{\alpha}{2}\|u(\cdot,t)\|_{L^2}^2+Im(u_t,u),\\ E(t)&=\|u_t(\cdot,t)\|_{L^2}^2+\|(-\Delta)^{\gamma/4}u(\cdot,t)\|_{L^2}^2+\frac{\beta}{2}\|u(\cdot,t)\|_{L^4}^4, \end{aligned}$$

with $Re(s)$ and $Im(s)$ represent the real and imaginary parts of complex number s,

$$\|u_t\|_{L^2}^2=\int_{\mathbb{R}}|u_t|^2dx, \quad \|u\|_{L^p}^p=\int_{\mathbb{R}}|u|^p dx, \quad p=2,4,$$

are the mass and energy, respectively.

As shown in [28] that the nonconservative schemes for Schrödinger-type equations may lead to numerical blow-up, thus, the conservative and unconditionally stable schemes become very important for solving Schrödinger and Schrödinger-type equations.

In the past several years, various conservative and accurate numerical methods have been developed for the NLSW, including spectral methods [15,23], finite element methods [5,9], finite difference methods [2,6,16,17,25,29,30] and so on. For the FNLSW, to the authors' knowledge, the literature limited. For instance, in [20], a linearly implicit conservative scheme is constructed based on the finite difference method. The Galerkin finite element method is used to solve the FNLSW in [14]. These limitations motivate us to develop an efficient and conservative scheme based on a high-order compact difference method and matrix transform technique (MTT) [10]. By virtue of MTT, one can efficiently