J. Math. Study
doi: 10.4208/jms.v54n4.21.09

Vol. **54**, No. 4, pp. 451-459
December 2021

Weakly I-semiregular Rings and I-semiregular Rings

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Received December 27, 2019; Accepted October 8, 2020;
Published online June, 29 2021.

Abstract. Let I be an ideal of a ring R. We call R weakly I-semiregular if R/I is a von Neumann regular ring. This definition generalizes I-semiregular rings. We give a series of characterizations and properties of this class of rings. Moreover, we also give some properties of I-semiregular rings.

AMS subject classifications: 16D50, 16E50

Key words: Weakly I-semiregular rings, I-semiregular rings, n-injective modules, n-flat modules, (m,n)-injective modules.

1 Introduction

Throughout this paper, m,n are positive integers, R is an associative ring with identity, I is an ideal of R, $J=J(R)$ is the Jacobson radical of R and all modules considered are unitary.

Recall that a ring R is called *semiregular* [11], if for any $a\in R$, there exists $e^2=e\in aR$ such that $(1-e)a\in J$. By [11, Theorem 2.9], a ring R is semiregular if and only if R/J is von Neumann regular and idempotents can be lifted modulo J. In [12], Nicholson and Yousif extend the concept of semiregular rings to *I-semiregular rings*. Let I be an ideal of R. Then following [12], an element $a\in R$ is called *left I-semiregular* if there exists $e^2=e\in Ra$ such that $a(1-e)\in I$, equivalentely if there exists $f^2=f\in aR$ such that $(1-f)a\in I$; a ring R is called left I-semiregular if every element of R is I-semiregular. It is easy to see that I-semiregular rings are left-right symmetric. I-semiregular rings have been studied by many authors (see, for example [2,12,13,16,17,20]). By [12, Theorem 1.2] or [13, Theorem 28], we see that if R is left I-semiregular, then R/I is regular and idempotents can be lifted modulo I.

In this article, we extend the concept of I-semiregular rings to weakly I-semiregular rings. Let I be an ideal of R. We will call R weakly I-semiregular if R/I is regular. A

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series of characterizations and properties of this class of rings will be given, and some properties of I-semiregular rings will be given too.

For any module M, M^+ denotes $\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$, where $\mathbb{Q}$ is the set of rational numbers, and $\mathbb{Z}$ is the set of integers. In general, for a set S, we write $S^{m\times n}$ for the set of all formal $m\times n$ matrices whose entries are elements of S, and S_n (resp., S^n) for the set of all formal $n\times 1$ (resp., $1\times n$) matrices whose entries are elements of S. Let M be a left R-module, $X\subseteq M_n$ and $A\subseteq R^{m\times n}$. Then we define $\mathbf{r}_{M_n}(A)=\{u\in M_n : au=0,\forall a\in A\}$, and $\mathbf{l}_{R^{m\times n}}(X)=\{a\in R^{m\times n} : ax=0,\forall x\in X\}$.

2 Weakly I-semiregular rings

At the begin of this section, we introduce the concept of weakly I-semiregular rings as following.

Definition 2.1. *Let I be an ideal of a ring R. Then R is said to be weakly I-semiregular if R/I is a von Neumann regular ring.*

Let T be an ideal of a ring R. Then following [13], we say that idempotents lift strongly modulo T, if $a^2-a\in T$, then there exists $e^2=e\in aRa$ such that $e-a\in T$.

Example 2.1. (1) By [13, Theorem 28], a ring R is I-semiregular if and only if it is weakly I-semiregular and idempotents lift strongly modulo I. In particular, a ring R is a semiregular ring and only if it is a weakly J(R)-semiregular and idempotents lift strongly modulo J(R).

(2) By [17, Theorem 1.6], a ring R is S_r-semiregular if and only if it is weakly S_r-semiregular, where $S_r=Soc(R_R)$.

(3) By [13, Example 24], there exists a commutative ring R which contains an ideal I such that $R/I\cong\mathbb{Z}_6$, but idempotents do not all lift modulo I. Note that the ring $\mathbb{Z}_6$ is von Neumann regular, so R is weakly I-semiregular but it is not I-semiregular. Also, $\mathbb{Z}$ is weakly I-semiregular for each non-zero semiprimitive ideal I, but not I-semiregular.

Let M be an R-module and N a submodule of M. According to [25], N is said to have a weak supplement L in M if $N+L=M$ and $N\cap L\ll M$, and M is called weakly supplemented if every submodule N of M has a weak supplement. It is easy to see that ${}_RR$ is weakly supplemented if and only if for any left ideal L of R, there is a left ideal K such that $L+K=R$ and $L\cap K\subseteq J(R)$. Inspired by this result, we have the following definition.

Definition 2.2. *Let I be an ideal of a ring R. Then a left ideal L of R is said to be I-weak supplemented in ${}_RR$ if there exists a left ideal K such that L+K=R and $L\cap K\subseteq I$. In this case, we call K an I-weak supplement of L in R.*

Similarly , we can define the concept of I-weak supplement of a right ideal.