Commun. Math. Res. Vol. **37**, No. 4, pp. 462-483
doi: 10.4208/cmr.2021-0012 November 2021

# The L* Partial Order on the Set of Group Matrices

Xiaoji Liu, Fang Gui and Hongxing Wang*

*School of Mathematics and Physics, Guangxi University for Nationalities, Nanning 530006, China.*

Received 27 January 2021; Accepted 9 June 2021

Dedicated to Roger Penrose for his 90th birthday

**Abstract.** In this paper, we use the Löwner partial order and the star partial order to introduce a new partial order (denoted by "L*") on the set of group matrices, and get some characteristics and properties of the new partial order. In particular, we prove that the L* partial order is a special kind of the core partial order and it is equivalent to the star partial order under some conditions. We also illustrate its difference from other partial orders with examples and find out under what conditions it is equivalent to other partial orders.

**AMS subject classifications**: 15A09, 15A57, 15A24

**Key words**: L* partial order, star partial order, core partial order, Löwner partial order, group matrix.

## 1 Introduction

In this paper, we use the following notations. The symbol $\mathbb{C}_{m,n}$ denotes the set of $m\times n$ matrices with complex entries; $\mathbb{H}(n)$, $\mathbb{H}_{\geq}(n)$, and $\mathbb{H}_{>}(n)$ denote the set of $n\times n$ Hermitian matrices, Hermitian nonnegative definite matrices, and Hermitian positive definite matrices, respectively. The symbols $A^*$, $\mathcal{R}(A)$, and $\mathrm{rk}(A)$ represent the conjugate transpose, range space, and rank of $A\in\mathbb{C}_{m,n}$, respectively. Let $A\in\mathbb{C}_{n,n}$, then the smallest positive integer $k$, where $\mathrm{rk}(A^{k+1})=\mathrm{rk}(A^k)$,

*Corresponding author. *Email addresses:* xiaojiliu72@126.com (X. Liu), 939978267@qq.com (F. Gui), winghongxing0902@163.com (H. Wang)

is called the index of $A$ and is denoted by $\mathrm{Ind}(A)$. The symbol $\mathbb{C}_n^{\mathrm{CM}}$ stands for the set of all group matrices of $\mathbb{C}_{n,n}$

$$\mathbb{C}_n^{\mathrm{CM}}=\left\{A|\mathrm{rk}(A^2)=\mathrm{rk}(A),\ A\in\mathbb{C}_{n,n}\right\}.$$

The Moore-Penrose inverse of $A\in\mathbb{C}_{m,n}$ is defined as the unique matrix $X\in\mathbb{C}_{n,m}$ satisfying the following equations:

$$\begin{array}{ll}(1)\ \ AXA=A, & (2)\ \ XAX=X,\\ (3)\ \ (AX)^*=AX, & (4)\ \ (XA)^*=XA,\end{array}$$

and is usually denoted by $X=A^\dagger$ [12]. The group inverse of $A\in\mathbb{C}_n^{\mathrm{CM}}$ is defined as the unique matrix $X\in\mathbb{C}_{n,n}$ satisfying the above relations (1), (2) and the equation

$$(5)\ \ AX=XA.$$

Such a matrix $X$ is usually denoted by $A^\#$ [12]. The core inverse of $A\in\mathbb{C}_n^{\mathrm{CM}}$ is defined as the unique matrix $X\in\mathbb{C}_{n,n}$ satisfying the above relations (1), (3) and the equation

$$(2^{'})\ \ AX^2=X.$$

Such a matrix $X$ is usually denoted by $A^{\textcircled{\#}}$ [13]. A matrix $A\in\mathbb{C}_{n,n}$ is called an $EP$ (or range-Hermitian) matrix if $\mathcal{R}(A)=\mathcal{R}(A^*)$. It is well-known that an $EP$ matrix is core-invertible, and the core inverse, Moore-Penrose inverse, and group inverse of the matrix are identical.

**Lemma 1.1** ([3])**.** *Given $A\in\mathbb{C}_n^{\mathrm{CM}}$ and $\mathrm{rk}(A)=r$, then there exists a unitary matrix $\widehat{U}$, by which*

$$A=\widehat{U}\begin{bmatrix}T & S\\ 0 & 0\end{bmatrix}\widehat{U}^*, \tag{1.1}$$

*where $T\in\mathbb{C}_{r,r}$ is nonsingular. Furthermore,*

$$A^\#=\widehat{U}\begin{bmatrix}T^{-1} & T^{-2}S\\ 0 & 0\end{bmatrix}\widehat{U}^*,\quad A^{\textcircled{\#}}=\widehat{U}\begin{bmatrix}T^{-1} & 0\\ 0 & 0\end{bmatrix}\widehat{U}^*,$$

*and*

$$A^\#=(A^{\textcircled{\#}})^2A,\quad A^2(A^{\textcircled{\#}})=(A^{\textcircled{\#}})^\#. \tag{1.2}$$

A binary relation is called a partial order if it is reflexive, transitive, and anti-symmetric on a non-empty set. In recent years, more and more mathematicians have turned their attention to matrix partial orders. The well-known partial orders are the minus, Löwner, core, star, C-N, GL, CL partial orders, etc. They are defined as (1)-(8) in the following (see [3,6–11,14]):