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The Dirichlet Problem for Stochastic Degenerate Parabolic-Hyperbolic Equations

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Abstract. We consider the Dirichlet problem for a quasilinear degenerate parabolic stochastic partial differential equation with multiplicative noise and non-homogeneous Dirichlet boundary condition. We introduce the definition of kinetic solution for this problem and prove existence and uniqueness of solutions. For the uniqueness of kinetic solutions we prove a new version of the doubling of variables method and use it to deduce a comparison principle between solutions. The proof requires a delicate analysis of the boundary values of the solutions for which we develop some techniques that enable the usage of the existence of normal weak traces for divergence measure fields in this stochastic setting. The existence of solutions, as usual, is obtained through a two-level approximation scheme consisting of nondegenerate regularizations of the equations which we show to be consistent with our definition of solutions. In particular, the regularity conditions that give meaning to the boundary values of the solutions are shown to be inherited by limits of nondegenerate parabolic approximations provided by the vanishing viscosity method.

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1 Introduction

Let $\mathcal{O}$ be a bounded smooth open subset of $\mathbb{R}^d$. We consider the Dirichlet problem for a quasilinear degenerate parabolic stochastic partial differential equation

$$du+\operatorname{div}\big(\mathbf{A}(u)\big)\,dt=D^2:\mathbf{B}(u)\,dt+\Phi(u)\,dW,\quad x\in\mathcal{O},\quad t\in(0,T), \tag{1.1}$$

$$u(0)=u_0, \tag{1.2}$$

$$u(t)\big|_{\partial\mathcal{O}}=u_b(t), \tag{1.3}$$

where, $\mathbf{A}:\mathbb{R}\to\mathbb{R}^d$ and $\mathbf{B}:\mathbb{R}\to\mathbb{M}^d$ are smooth maps, $\mathbb{M}^d$ denote the space of $d\times d$ matrices. For $\mathbf{R}=(R_{ij}),\mathbf{S}=(S_{ij})\in\mathbb{M}^d$ we denote $\mathbf{R}:\mathbf{S}:=\sum_{i,j}R_{ij}S_{ij}$ and, by extrapolation, $D^2:\mathbf{B}:=\sum_{i,j}\partial^2_{x_ix_j}B_{ij}$. The matrix $\mathbf{B}(u)$ is symmetric and its derivative $\mathbf{b}(u)=\mathbf{B}'(u)$ is a symmetric nonnegative $d\times d$ matrix. W is a cylindrical Wiener process.

1.1 Hypotheses

The flux function $\mathbf{A}=(A_1,\cdots,A_d):\mathbb{R}\to\mathbb{R}^d$ is assumed to be of class C^2 and we denote its derivative by $\mathbf{a}=(a_1,\cdots,a_d)$. The diffusion matrix $\mathbf{b}=(b_{ij})_{i,j=1}^d:\mathbb{R}\to\mathbb{M}^d$ is symmetric and positive semidefinite. Its square-root matrix, also symmetric and positive semidefinite, is denoted by σ, which is assumed to be bounded and locally γ-Hölder continuous for some $\gamma>1/2$, that is,

$$|\sigma(\xi)-\sigma(\zeta)|\le C(R)|\xi-\zeta|^\gamma\quad\text{for all}\quad \xi,\zeta\in\mathbb{R},\quad |\xi-\zeta|<R. \tag{1.4}$$

Moreover, we assume that, for some $b\in C^1(\mathbb{R})$ with $b'(u)\ge 0$ for all $u\in\mathbb{R}$, and a constant $\Lambda>1$ we have

$$b'(u)^2|\xi|^2\le\sum_{i,j}b_{ij}(u)\xi_i\xi_j\le\Lambda b'(u)^2|\xi|^2. \tag{1.5}$$