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Data Recovery from Cauchy Measurements in Transient Heat Transfer

Thouraya Baranger Nouri[1,*] and
Faker Ben Belgacem[2]

[1] *Université de Lyon, Université Lyon 1, LMC2, EA 7427, F69622 Villeurbanne Cedex, France.*
[2] *Université de Technologie de Compiègne, LMAC, EA 2222, BP 20529, 60205 Compiègne cedex, France.*

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Abstract. We study the ill-posedness degree of the reconstruction processes of missing boundary data or initial states in the transient heat conduction. Both problems are severely ill-posed. This is a powerful indicator about the way the instabilities will affect the computations in the numerical recovery methods. We provide rigorous proofs of this result where the conductivites are space dependent. The theoretical work is concerned with the unsteady heat equation in one dimension even though most of the results obtained here are readily extended to higher dimensions.

AMS subject classifications: 52B10, 65D18, 68U05, 68U07.

Key words: Data completion process, ill-posedness degree, Cauchy matrix, convolution equations, parabolic regularity.

1 Introduction

Computational recovery processes of missing boundary data or initial states from Cauchy measurements in transient heat transfer seem recurrent in many areas in sciences and engineering (see [5, 6, 23]). They turn out to be among few pertinent ways to proceed, if not the only one, when engineers are interested for example in quantifying front surface heat inputs of a (thin) plate from back surface outputs. Mounting measurements set-up along the front surface may not be feasible due to harsh environmental conditions. Practitioners are therefore led to place sensors at the back surface; they are then left with the evaluation, by means of affordable (analytical and/or numerical) tools, of the heat

*Corresponding author. *Email addresses:* `thouraya.baranger@univ-lyon1.fr` (Baranger Nouri T), `faker.ben-belgacem@utc.fr` (Ben Belgacem F)

transfer up to the front surface. The particularity of the related mathematical problem is the ill-posedness. Missing data to reconstruct suffer from high instabilities generated by unavoidable perturbations affecting the available data because of the finite accuracy of measuring instruments (see [2,3,14,17,18,22,31,32]). As a consequence, computing methods crudely employed for numerical handling of these inverse problems are most often doomed to fail unless they are used with relevant regularization techniques combined to suitable automatic selection rules of the regularization parameter(s). Readers are referred to [4,15,19,24,25,33] for a general exposition of these issues. We are exclusively focused on the issue of ill-posedness degree for the reconstruction problem of either the initial or the boundary data from Cauchy's measurements. The purpose is then to perform a rigorous analysis on the severe ill-posed of the inverse problems under scope. We develop in details methodologies for the transient heat equation set in a rod. This choice is made only for seek of simplicity. We do not see why the central ideas discussed here cannot be effective for higher dimensions.

The contents of the paper are as follows. Section 2 is dedicated to the study of the initial state recovery from Cauchy's conditions. After defining the linear operator to invert, we study some of its marked features. Expanding this operator along Fourier basis shows that the ill-posedness degree is connected with a Cauchy matrix. This matrix is spectrally equivalent to a Pick matrix. Using the theory elaborated in [7], we exhibit the asymptotics of its eigenvalues. They decay exponentially fast towards zero which is an indication of the severe ill-posedness of the reconstruction process of the initial state. In Section 3, we turn to the recovery of a missing boundary condition at one extremity of the rod, where the initial state is known. We follow a similar approach, we define the linear operator to analyze and underline its distinctive properties. The key point is to put it under an integral form using a suitable Green kernel. The smoothness and the flatness at the origin of that kernel is the clue to the analysis of the ill-posedness degree. It is derived in the appendix thanks to the Laplace transform and some comparison results. In Section 4, we provide two numerical illustrations by MATLAB of the technical results stated in the previous sections.

Notation 1.1. Let X be a Banach space endowed with its norm $\|\cdot\|_X$. We denote by $L^2(0,T;X)$ the space of measurable functions v from $(0,T)$ in X such that

$$\|v\|_{L^2(0,T;X)} = \left(\int_{(0,T)} \|v(s)\|_X^2 ds\right)^{1/2} < +\infty.$$

We also use the space $\mathscr{C}(0,T;X)$ of continuous functions v from $[0,T]$ in X. Denote by I a given interval in $\mathbb{R}$, the Sobolev space $H^1(I)$ is the space all the functions that belong to $L^2(I)$ together with their first derivatives.