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Vector Fields of Cancellation for the Prandtl Operators

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Abstract. It has been a fascinating topic in the study of boundary layer theory about the well-posedness of Prandtl equation that was derived in 1904. Recently, new ideas about cancellation to overcome the loss of tangential derivatives were obtained so that Prandtl equation can be shown to be well-posed in Sobolev spaces to avoid the use of Crocco transformation as in the classical work of Oleinik. This short note aims to show that the cancellation mechanism is in fact related to some intrinsic directional derivative that can be used to recover the tangential derivative under some structural assumption on the fluid near the boundary.

AMS subject classifications: 35Q30, 35Q31

Key words: Prandtl operators, cancellation mechanism, vector field of cancellation, well-posedness theory, structural assumptions.

1 Introduction

In 1904, Prandtl derived the famous equation to describe the fluid behaviour near a boundary by resolving the difference between the viscous and the inviscid effects with no-slip boundary condition. This revolutionary result has vast applications in aerodynamics and other areas of engineering. It also provides a typical mathematical model that attracts attention even now because a lot of mathematical problems remain unsolved. The key observation by Prandtl is that outside

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a layer of thickness of $\sqrt{1/\mathrm{Re}}$, convection dominates so that the flow is governed by the Euler equations; while inside a layer (boundary layer) of thickness of $\sqrt{1/\mathrm{Re}}$, convection and viscosity balance so that the flow is governed by the Prandtl equations. Here Re is the Reynolds number.

Let us briefly recall the derivation of the Prandtl equation. Consider the incompressible Navier-Stokes equations over a flat boundary $\{(x,y)\in D, z=0\}$ with no-slip boundary condition,

$$\begin{cases} \partial_t \mathbf{u}^\epsilon + (\mathbf{u}^\epsilon \cdot \nabla)\mathbf{u}^\epsilon + \nabla p^\epsilon - \epsilon\mu\Delta\mathbf{u}^\epsilon = 0, \\ \nabla \cdot \mathbf{u}^\epsilon = 0, \\ \mathbf{u}^\epsilon|_{z=0} = 0, \end{cases}$$

where $\mathbf{u}^\epsilon$ is the velocity field, p^ϵ represents the pressure and $\epsilon\mu$ is the viscosity coefficient with ϵ being a small parameter. According to the Prandtl ansatz, set $\mathbf{u}^\epsilon = (u^\epsilon, v^\epsilon, w^\epsilon)^T$ with the following scaling:

$$\begin{cases} u^\epsilon(t,x,y,z) = u\left(t,x,y,\dfrac{z}{\sqrt{\epsilon}}\right) + o(1), \\ v^\epsilon(t,x,y,z) = v\left(t,x,y,\dfrac{z}{\sqrt{\epsilon}}\right) + o(1), \\ w^\epsilon(t,x,y,z) = \sqrt{\epsilon}w\left(t,x,y,\dfrac{z}{\sqrt{\epsilon}}\right) + o(\sqrt{\epsilon}). \end{cases}$$

The leading order gives the following classical Prandtl equations:

$$\begin{cases} \partial_t u + (u\partial_x + v\partial_y + w\partial_z)u + \partial_x p^E(t,x,y,0) = \mu\partial_z^2 u, \\ \partial_t v + (u\partial_x + v\partial_y + w\partial_z)v + \partial_y p^E(t,x,y,0) = \mu\partial_z^2 v, \\ \partial_x u + \partial_y v + \partial_z w = 0, \\ (u,v,w)|_{z=0} = 0, \quad \lim\limits_{z\to+\infty}(u,v) = \left(u^E, v^E\right)(t,x,y,0), \end{cases}$$

where the fast variable $z/\sqrt{\epsilon}$ is still denoted by z for simplicity of notation. And the pressure and velocity of the outer flow denoted by $p^E(t,x,y)$ and $\mathbf{u}^E = (u^E, v^E, 0)$ $\times(t,x,y)$ satisfy the Bernoulli's law

$$\partial_t \mathbf{u}^E + \left(\mathbf{u}^E \cdot \nabla\right)\mathbf{u}^E + \nabla p^E = 0.$$

For later presentation, we denote the Prandtl operator by

$$P^\mu = \partial_t + u\partial_x + v\partial_y + w\partial_z - \mu\partial_z^2$$

with a parameter μ in front of the dissipation in the normal direction.