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Greedy Kaczmarz Algorithm Using Optimal Intermediate Projection Technique for Coherent Linear Systems

Fang Geng, Li-Xiao Duan and Guo-Feng Zhang*

School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China

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Abstract. The Kaczmarz algorithm is a common iterative method for solving linear systems. As an effective variant of Kaczmarz algorithm, the greedy Kaczmarz algorithm utilizes the greedy selection strategy. The two-subspace projection method performs an optimal intermediate projection in each iteration. In this paper, we introduce a new greedy Kaczmarz method, which give full play to the advantages of the two improved Kaczmarz algorithms, so that the generated iterative sequence can exponentially converge to the optimal solution. The theoretical analysis reveals that our algorithm has a smaller convergence factor than the greedy Kaczmarz method. Experimental results confirm that our new algorithm is more effective than the greedy Kaczmarz method for coherent systems and the two-subspace projection method for appropriate scale systems.

AMS subject classifications: 15A06, 65F10, 65F20, 65F25, 65F50

Key words: Kaczmarz algorithm, two-subspace, greedy methods, linear systems.

1. Introduction

We are going to solve the consistent linear system

$$Ax = b, \tag{1.1}$$

where the matrix $A \in \mathbb{R}^{m\times n}$, the vector $b \in \mathbb{R}^m$. Nowadays, these linear systems (1.1) mainly come from computed tomography [12], signal processing [15, 17], machine learning [4, 25] and many other fields. The Kaczmarz algorithm [13], one of the most

*Corresponding author. *Email addresses:* duanlx17@lzu.edu.cn (L.-X. Duan), gengf19@lzu.edu.cn (F. Geng), gf_zhang@lzu.edu.cn (G.-F. Zhang)

popular solver for such system (1.1), is an iterative method which uses a small piece of information in each iteration. The method was first proposed in the 1930s and was used to deal with image reconstructions problems under the name algebraic reconstruction technology (ART) in the 1970s.

The Karczmarz method starts from a given initial guess x_0, then iteratively projects the approximation x_k which is generated in the k-step onto the solution space of a single equation. Subsequently, about ten years ago, Strohmer and Vershynim [27] creatively developed and analyzed a randomized Kaczmarz (RK) method, which quickly attracted widespread attention with its simple iterative format. At each iteration, RK method randomly selects the row index r_k from the set $[1, m]$ with probability $\frac{\|a_{r_k}\|_2^2}{\|A\|_F^2}$, and gets the updated solution x_k by using the information of equation $a_{r_k}^{\mathrm{T}} x = b_{r_k}$. The format can be simply described as

$$x_k = x_{k-1} + \frac{b_{r_k} - a_{r_k}^{\mathrm{T}} x_{k-1}}{\|a_{r_k}\|_2^2} a_{r_k},$$

where $a_{r_k}^{\mathrm{T}}$ is the r_k-th row of matrix A, b_{r_k} is the r_k-th entry of vector b. Some variants and extensions of this algorithm have been proposed one after another, such as extending RK to solve inconsistent linear systems [31], modified to block Kaczmarz method [21, 23], improving to the average block Kaczmarz method [8, 19, 20], and so on [9, 10, 16].

Greedy Kaczmarz (GK) is a popular extended Kaczmarz method which use the greedy selection strategy for r_k. In [24], the authors analyzed the convergence rate for GK. Recently, Bai and Wu [2, 3] researched variants of GK in which r_k was randomly selected according to a specified sample distribution related to the maximal residual value, and named variants greedy randomized Kaczmarz (GRK) and relaxed greedy randomized Kaczmarz (RGRK), respectively. Soon after, inspired by their proof, Du and Gao [7] provided a convergence analysis for a generalized version of GK which is the maximal weighted residual Kaczmarz (MWRK) algorithm [18]. In [30], the author proposed a geometric probability randomized Kaczmarz (GPRK) method which is similar to the work of Bai and Wu. In [6], the authors proposed a new solver establishing the connection between Motzkin's relaxation method (MM) and RK, named the Sampling Kaczmarz-Motzkin (SKM) method. Hereafter, the authors given an improved analysis of SKM in [11]. There are some other improvements such as [14, 26].

There is another useful improvement of RK which was proposed by Needell and Ward [22]. The improved algorithm was called the two-subspace projection (TSP) method which required two rows of the system for each iteration and utilized an optimal intermediate projection technique to accelerate RK. Soon after, Wallace and Sekmen [28] introduced subspace orthogonal projections approach on this basis. Recently, an extended TSP method was put forward to solve inconsistent linear systems in [29].

For the remainder of this section, we introduce some notations. We use $\|\cdot\|_2$ to represent the Euclidean norm. For any matrix $A \in \mathbb{R}^{m\times n}$, $\sigma(A)$ and $\lambda(A)$ are defined