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The Obstacle Problem For Nonlinear Degenerate Elliptic Equations with Variable Exponents and L^1-Data

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Abstract. The aim of this paper is to study the obstacle problem associated with an elliptic operator having degenerate coercivity, and L^1-data. The functional setting involves Lebesgue-Sobolev spaces with variable exponents. We prove the existence of an entropy solution and show its continuous dependence on the L^1-data in $W^{1,q(\cdot)}(\Omega)$ with some $q(\cdot)>1$.

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1 Introduction

Let $\Omega\subset\mathbb{R}^N$ $(N\geq 2)$ be a bounded domain with smooth boundary $\partial\Omega$. and $f\in L^1(\Omega)$. We consider the following nonlinear problem

$$\begin{cases} Au=f, & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where A the operateur define by

$$Au=-\text{div}\frac{a(x,\nabla u)}{(1+b(x)|u|)^{\gamma(x)}}. \tag{1.2}$$

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$p,\gamma \in C(\overline{\Omega})$, and $p^+ := \max_{x\in\overline{\Omega}} p(x)$, $p^- := \min_{x\in\overline{\Omega}} p(x)$; furthermore, p,γ satisfy

$$2-\frac{1}{N} < p^- \le p(x) \le p^+ < N, \quad \forall x \in \overline{\Omega}, \tag{1.3}$$

$$0 \le \gamma^+ < \min\left\{ \frac{p(x)-1}{N-p(x)+1}, \frac{N(p(x)-1)}{N-1} - 1 \right\}, \quad \gamma^+ < p^- - 1, \tag{1.4}$$

and b is an $L^\infty-$function satisfying, with some $\nu \ge 0$

$$0 \le b(x) \le \nu, \tag{1.5}$$

$a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function, i.e. $a(.,\xi)$ is measurable in Ω, for any ξ in $\mathbb{R}^N$; and $a(x,.)$ is continuous in $\mathbb{R}^N$, for almost every $x \in \Omega$. Meanwhile, a satisfies

$$a(x,\xi).\xi \ge \alpha |\xi|^{p(x)}, \tag{1.6}$$

$$|a(x,\xi)| \le \beta\Big(j(x) + |\xi|^{p(x)-1} \Big), \tag{1.7}$$

$$(a(x,\xi)-a(x,\eta))(\xi-\eta) > 0, \tag{1.8}$$

$$|a(x,\xi)-a(x,\zeta)| \le \mu \begin{cases} |\xi-\zeta|^{p(x)-1}, & \text{if } 1 < p(x) < 2, \\ (1+|\xi|+|\zeta|)^{p(x)-2}|\xi-\zeta|, & \text{if } p(x) \ge 2, \end{cases} \tag{1.9}$$

for almost every $x \in \Omega$ and for every $\xi,\eta,\zeta \in \mathbb{R}^N$ with $\xi \ne \eta$, where α,β,μ are constants, and j is a nonnegative function in $L^{p'(\cdot)}(\Omega)$.

If f has a fine regularity, e.g., $f \in W^{-1,p'(\cdot)}(\Omega)$, the obstacle problem corresponding to (f,ψ,g) can be formulated in terms of the inequality

$$\int_\Omega \frac{a(x,\nabla u)}{(1+b(x)|u|)^{\gamma(x)}}.\nabla(u-v)dx + \int_\Omega f(u-v)dx \ge 0, \tag{1.10}$$

for every $v \in K_{g,\psi} \cap L^\infty(\Omega) \ge 0$, whenever the convex subset

$$K_{g,\psi} = \left\{ v \in W^{1,p(\cdot)}(\Omega);\, v-g \in W_0^{1,p(\cdot)}(\Omega), v \ge \psi, a.e. \text{ in } \Omega \right\} \ne \emptyset$$

is nonempty. However, if $f \in L^1(\Omega)$, the right integral in (1.10) is not well-defined.

Following [1] etc., we are led to the more general definition of a solution to the obstacle problem, using the truncation function at level $k>0$, $T_k : \mathbb{R} \to \mathbb{R}$ defined by

$$T_k(s) = \begin{cases} s, & \text{if } |s| \le k, \\ k\frac{s}{|s|}, & \text{if } |s| > k. \end{cases}$$