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SOLVABILITY FOR FRACTIONAL FUNCTIONAL DIFFERENTIAL EQUATION BOUNDARY VALUE PROBLEMS AT RESONANCE*†

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Abstract

The paper deals a fractional functional boundary value problems with integral boundary conditions. Besed on the coincidence degree theory, some existence criteria of solutions at resonance are established.

Keywords fractional boundary value problem; at resonance; coincidence degree theory; integral boundary conditions

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1 Introduction

This paper deals the following fractional functional differential equation boundary value problems at resonance

$$\begin{cases} D_{0+}^{\alpha}x(t) = f(t, x_t), \quad t \in (0,1), \\ x(0) = x'(0) = x''(0) = 0, \\ x(1) = \mu \displaystyle\int_0^{\eta} x(s)\mathrm{d}s, \\ x(t) = \xi(t), \quad t \in [-\tau, 0] \end{cases} \tag{1.1}$$

with $3 < \alpha \le 4$, $\frac{\mu\eta^{\alpha}}{\alpha} = 1$, $x_t(s) = x(t+s)$ for $t \in [0,1]$, $s \in [-\tau, 0]$, D_{0+}^{α} is the Riemann-Liouville fractional derivative, $0 < \tau < \eta < 1$. Let $C_\tau = C[-\tau, 0]$ with the norm $\|x\|_{[-\tau,0]} = \max\limits_{t\in[-\tau,0]} |x(t)|$, $\xi \in C_\tau$.

Fractional derivative was introduced by Leibnitz in the email to L'Hospital [1]. It was not developed before the 20th century, since it was short of a physical meaning or application. In recent decades, the researchers have found that the fractional

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derivative has long-term memory and self-similarity so that it can be used in electromagnetic, viscoelasticity, and other fields [2-4]. Motivated by the widely application of fractional derivative, the fractional differential equations have received a lot of attention. There are a lot of papers dealing with the solutions for fractional differential equation boundary value problems [5-13].

Zhang, Lin and Sun [13] considered the following boundary value problem

$$\begin{cases} D_{0+}^{\alpha}u(t) = h(t)f(t,u(t)), \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, \\ u(1) = \lambda \displaystyle\int_0^{\eta} u(s)\mathrm{d}s, \end{cases} \tag{1.2}$$

with $0 \leq \frac{\lambda\eta^{\alpha}}{\alpha} < 1$. If $\frac{\lambda\eta^{\alpha}}{\alpha} = 1$, problem (1.2) is at resonance, that is, its associated homogeneous problem

$$\begin{cases} D_{0+}^{\alpha}u(t) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, \\ u(1) = \lambda \displaystyle\int_0^{\eta} u(s)\mathrm{d}s \end{cases} \tag{1.3}$$

has a nontrivial solution $u(t) = ct^{\alpha-1}$, $c \in \mathbb{R}$. Hence the research method of [13] is not applicable to (1.1) at resonance. Inspired by the above works, we consider (1.1) at resonance in this paper.

2 Preliminary

Some definitions and lemmas are presented which are available in the proof of our main results.

Definition 2.1 The Riemann-Liouville fractional integral of order α for function f is defined as

$$I_{0+}^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\mathrm{d}s, \quad \alpha > 0,$$

provided that the right side is point-wise defined on $(0,\infty)$.

Definition 2.2 The Riemann-Liouville fractional derivative of order $\alpha > 0$ for function f is defined as

$$D_{0+}^{\alpha}y(t) = \frac{1}{\Gamma(n-\alpha)}\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}}\mathrm{d}s, \quad \alpha > 0,$$

where $n = [\alpha] + 1$, provided that the right side is point-wise defined on $(0,\infty)$.

Lemma 2.1[8] *Let $\alpha > 0$ and assume that $u \in C(0,1) \cap L(0,1)$, then the fractional differential equation*

$$D_{0+}^{\alpha}u(t) = 0$$