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On the Main Aspects of the Inverse Conductivity Problem

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Abstract We consider a nonlinear inverse problem for an elliptic partial differential equation known as the Calderón problem or the inverse conductivity problem. Based on several results, we briefly summarize them to motivate this research field. We give a general view of the problem by reviewing the available results for C^2 conductivities. After reducing the original problem to the inverse problem for a Schrödinger equation, we apply complex geometrical optics solutions to show its uniqueness. After extending the ideas of the uniqueness proof result, we establish a stable dependence between the conductivity and the boundary measurements. By using the Carleman estimate, we discuss the partial data problem, which deals with measurements that are taken only in a part of the boundary.

Keywords Calderón problem, Inverse conductivity problem, Dirichlet-to-Neumann map, Complex geometrical optics solutions, Carleman estimate.

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1. Introduction

You may ask what the inverse conductivity problem is. Well, to answer this question, as the name of the problem indicates, we should consider the direct conductivity problem first, given by

$$\begin{cases} \nabla \cdot \gamma \nabla w = 0 \text{ in } \Omega, \\ \qquad w = f \text{ on } \partial\Omega, \end{cases}$$

where Ω is a bounded open set of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, $\gamma \in C^2(\bar{\Omega})$ is a positive real-valued function that represents the electrical conductivity of the domain Ω. Physically interpreted, the application of a voltage $f \in H^{1/2}(\partial\Omega)$ on the boundary induces an electrical potential w in the interior of Ω, where $w \in H^1(\Omega)$ is the unique weak solution of this elliptic boundary value problem.

We define the Dirichlet-to-Neumann map (DN map) Λ_γ by relating a boundary voltage f (Dirichlet data) to the flux at the boundary $\gamma \frac{\partial w}{\partial \nu}$ (Neumann data) as follows:

$$\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega),$$

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$$f \mapsto \Lambda_\gamma(f) = \gamma \frac{\partial w}{\partial \nu}\Big|_{\partial\Omega},$$

where $\frac{\partial}{\partial \nu}$ is the outward normal derivative at $\partial\Omega$.
From the variational formulation of the precedent problem, it is clear that

$$\langle \Lambda_\gamma f, g\rangle = \left\langle \gamma \frac{\partial w}{\partial \nu}, g \right\rangle = \int_\Omega \gamma \nabla w \nabla z dx \quad \forall f, g \in H^{1/2}(\partial\Omega),$$

where $z \in H^1(\Omega), z\big|_{\partial\Omega} = g$. It follows from this definition that Λ_γ is a bounded linear map from $H^{1/2}(\partial\Omega)$ into $H^{-1/2}(\partial\Omega)$. In this context, Calderón in his pioneer paper [9] formulated the Calderón problem as being the problem studying the inversion of the map $\gamma \mapsto \Lambda_\gamma$, i.e., the posed question is whether we can determine γ from the knowledge of $\Lambda_\gamma f$ in each $f \in H^{1/2}(\partial\Omega)$. This inversion method is also called electrical impedance tomography (EIT). It is a medical imaging technology with several applications, including the detection of breast cancer and pulmonary imaging. For more detailed arguments on this technique, see the review papers [6, 18].

The determination of γ from the DN map has different aspects. In this paper, we answer the preceding question in the interior of the studied domain by giving results on the three aspects: uniqueness, stability and partial data. For the boundary determination, in the case that smooth conductivities Kohn and Vogelius [21] proved that Λ_γ determines γ and all its normal derivatives on the boundary. More general results were shown in [2, 31]. In particular, Brown [7] proved that we could recover the boundary values of a $W^{1,1}$ or a C^0 conductivity from the knowledge of Λ_γ.

While the current paper only deals with the inverse conductivity problem in three and higher dimensions, we mention that the approach for the two-dimensional problem is quite different, which is essentially based on complex analysis. We refer readers to the work of Astala and Päivärinta [5] on bounded measurable conductivities for a deeper understanding of the problem in the plane.

In the following, we only consider isotropic conductivities, which are not dependent on direction. If a conductivity depends on direction, it is called an anisotropic conductivity. In this case, we are in the presence of the anisotropic Calderón problem. In the plane, uniqueness was shown for L^∞ anisotropic conductivities in [4]. For $n \geq 3$, this problem is also called Calderón's inverse problem on Riemannian manifolds, and as was pointed out in [23], this is a geometrical problem that has up to now remained open. For more detailed arguments, please also see [12] .

There are several problems related to the main one. The fractional Calderón problem is a nonlocal version of the classical one [11]. It was first introduced in [15]. In the present work, it is a question to study a Schrödinger operator containing an electrical potential. However, if there is also a nonzero magnetic potential, we are in the presence of another variant of the standard problem, namely the Calderón problem for the magnetic Schrödinger operator [22]. By combining the two precedent problems, we can also define another closely related one, which is the inverse conductivity problem for the fractional magnetic operator, and it is the subject of [24, 25].

Under the broad research field of the Calderón problem, we focus on its main aspects. We propose a simplified review of Salo's lecture notes [28] and some chapters from [13]. The rest of this article is organized in the following way: the applied notation and background knowledge are summarized in Section 2. In Section 3, we