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Existence Results for Fractional Differential Equations with the Riesz-Caputo Derivative

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Abstract In this paper, we apply some fixed point theorems to attain the existence of solutions for fractional differential equations with the space-time Riesz-Caputo derivative. We study the boundary value problems that the nonlinearity term f is relevant to fractional integral and fractional derivative. In addition, the boundary conditions involve integral. Two examples are given to show the effectiveness of theoretical results.

Keywords Fractional differential equation, Riesz-Caputo derivative, Fixed point theorem.

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1. Introduction

With the progress of modern science and technology and the advancement of fractional order theory, fractional order differential equations have been widely used in signal and image processing, electromagnetism, mechanics, optics and other fields of science and engineering. It is of profound significance to solve practical problems. See [1–3, 10–13, 15, 16] and the references therein.

Specially, there are few papers that studied the fractional differential equations problems with the Riesz-Caputo derivative. For the Riesz fractional derivative is a two-sided operator which holds memory effects. In [5], Chen et al. investigated the following equations:

$$\begin{cases} {}^{RC}_{0}D^{\gamma}_{T}y(\tau) = g(\tau, y(\tau)), \quad 0 \le \tau \le T, 0 < \gamma \le 1, \\ y(0) = y_0, \quad y(T) = y_T, \end{cases}$$

where ${}^{RC}D^{\gamma}$ is a Riesz-Caputo derivative and $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, y_0 and y_T are two constants. In [7], Gu et al. employed the Leray-Schauder and Krasnosel'skii fixed point theorems to show the existence of positive solutions for the above boundary value problems in [5], where $0 \le \tau \le 1$.

In [6], Chen et al. considered the anti-periodic boundary value problems:

$$\begin{cases} {}^{RC}_{0}D^{\gamma}_{T}y(\tau) = g(\tau, y(\tau)), \quad 0 \le \tau \le T, 1 \le \gamma \le 2, \\ y(0) + y(T) = 0, \quad y'(0) + y'(T) = 0, \end{cases}$$

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where $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. It reflected the future and the past nonlocal memory effects. In [2], Ahmad et al. investigated the existence of solutions for a nonlinear fractional integro-differential equation involving two Caputo fractional derivatives of different orders and a Riemann-Liouville integral, and equipped with dual anti-periodic boundary conditions. The authors introduced a new concept of dual anti-periodic boundary conditions.

In [14], the authors considered the following singular fractional boundary value problems of fractional differential equations:

$$\begin{cases} {}^C D^{\alpha}_{0+} u(t) = f(t, u(t), u'(t), {}^C D^{\beta}_{0+} u(t)), \\ u(0) + u(1) = 0, \quad u'(0) = 0, \end{cases}$$

where $f(t,x,y,z)$ is singular at the value 0 of its space variables x, y and z, $1 < \alpha < 2$, $0 < \beta < 1$, ${}^C D^{\theta}_{0+}$ is Caputo fractional derivative. By using the Vitali convergence theorems and fixed point theorem, the existence results of monotone solutions are attained.

In [4], the authors studied the existence of solutions for Caputo type sequential fractional integro-differential equations and inclusions:

$$\begin{aligned} &\left({}^C D^{\alpha} + \lambda {}^C D^{\alpha-1}\right) u(t) = f(t, u(t), {}^C D^{p} u(t), I^{q} u(t)), \qquad && t \in (0,1), \\ &\left({}^C D^{\alpha} + \lambda {}^C D^{\alpha-1}\right) u(t) \in F(t, u(t), I^{q} u(t)), && t \in (0,1), \end{aligned}$$

supplemented with the nonlocal boundary conditions

$$u(0) = h(u), u'(0) = u''(0) = 0, a I^{\beta} u(\xi) = \int_0^1 u(s) dH(s),$$

where ${}^C D^{\alpha}$ is the Caputo fractional derivative of order α, I^q is the Riemann-Liouville fractional integral of order q, $\alpha \in (3,4]$, $p \in (0,1)$, $\lambda > 0$, $\xi \in (0,1]$, $a \in \mathbb{R}$, $\beta > 0$, f is a nonlinear function, F is a nonlinear multivalued function. In [3], authors studied a new nonlocal boundary value problem of integro-differential equations involving mixed left and right Caputo and Riemann-Liouville fractional derivatives and Riemann-Liouville fractional integrals of different orders. The existence of solutions are obtained by using Leray-Schauder nonlinear alternative, Krasnosel'skii fixed point theorem and Banach contraction mapping principle.

Inspired by the works mentioned above, we study the existence and uniqueness of solutions of fractional differential equations with Riesz-Caputo derivative:

$$\begin{cases} {}^{RC}_{0} D^{\alpha}_{1} u(t) = f(t, u(t), {}_0 I^{\beta}_t u(t), {}^C_0 D^{\alpha-1}_t u(t)), \quad 0 \le t \le 1, \\ u(0) = 0, \quad u'(0) + u'(1) = 0, \quad u(1) = \displaystyle\int_0^1 u(t) dt, \end{cases} \tag{1.1}$$

where $1 < \alpha \le 2$, $\beta > 0$, ${}^{RC}D$ is a Riesz-Caputo derivative, ${}_0 I^{\beta}_t$ is the left Riemann-Liouville fractional integral of order β, ${}^C_0 D^{\alpha-1}_t$ is the left Caputo derivative of order $\alpha - 1$, and $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous function. By using the fixed point theorems, the existence results for fractional differential equations with the Riesz-Caputo derivative are obtained under some conditions.

Few literature studied the fractional differential equations with the Riesz-Caputo derivative. Compared with the existing literature [5–7], the new feature lying in this paper is that we investigate BVP(1.1), in which the nonlinearity term f is relevant