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Stability of Peakons for a Nonlinear Generalization of the Camassa-Holm Equation*

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Abstract In this paper, by using the dynamic system method and the known conservation laws of the gCH equation, and underlying features of the peakons, we study the peakon solutions and the orbital stability of the peakons for a nonlinear generalization of the Camassa-Holm equation (gCH). The gCH equation is first transformed into a planar system. Then, by the first integral and algebraic curves of this system, we obtain one heteroclinic cycle, which corresponds to a peakon solution. Moreover, we give a proof of the orbital stability of the peakons for the gCH equation.

Keywords Camassa-Holm equation, Peakon, Stability, Heteroclinic cycle, Orbital stability.

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1. Introduction

Due to its importance in breaking waves, the Camassa-Holm (CH) equation

$$u_t - u_{txx} - 3uu_x + 2u_x u_{xx} + uu_{xxx} = 0 \tag{1.1}$$

and related theories have been widely studied, see for instance [2–4, 6, 7, 11] and references therein. Recently, in [1], Anco and Recio have obtained single peakon and multi-peakon solutions to the following nonlinear generalization of the CH equation (gCH)

$$u_t - u_{xxt} = \frac{1}{2}(p+1)(p+2)u^p u_x - \frac{1}{2}p(p-1)u^{p-2}u_x^3 - 2pu^{p-1}u_x u_{xx} - u^p u_{xxx}, \tag{1.2}$$

where p is an arbitrary nonlinearity power. When $p = 1$, the gCH equation (1.2) becomes the CH equation (1.1) .

Similar to the CH equation, the gCH equation (1.2) also has the form of conservation law

$$m_t - \left(\frac{1}{2}pu^{p-1}\left(u^2 - u_x^2\right) + u^p m\right)_x = 0, \; m = u - u_{xx}. \tag{1.3}$$

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Thus, the integral

$$P=\int_{-\infty}^{+\infty} m dx \tag{1.4}$$

is conserved under the appropriately asymptotic decay condition on u.

The Hamiltonian

$$E_{(p)}=\int_{-\infty}^{+\infty} \frac{1}{2} u^{p}\left(u^{2}+u_{x}^{2}\right) dx, p \neq 0 \tag{1.5}$$

gives another conservation integral, which gives rise to the conservation law

$$D_{t}\left(\frac{1}{2} u^{p}\left(u^{2}+u_{x}^{2}\right)\right)+D_{x} X=0 \tag{1.6}$$

with

$$\begin{aligned} X= & -u^{p} u_{t} u_{x}+\frac{1}{2}\left(\left(1-D_{x}^{2}\right)^{-1}\left(\frac{1}{2} p u^{p-1}\left(u^{2}-u_{x}^{2}\right)+u^{p}\left(u-u_{x x}\right)\right)\right)^{2} \\ & -\frac{1}{2}\left(D_{x}\left(\left(1-D_{x}^{2}\right)^{-1}\left(\frac{1}{2} p u^{p-1}\left(u^{2}-u_{x}^{2}\right)+u^{p}\left(u-u_{x x}\right)\right)\right)\right)^{2} . \end{aligned} \tag{1.7}$$

The classification of conservation laws of the gCH equation is based on both [1] and [22]. When $p=3$, equation (1.2) becomes

$$u_{t}-u_{t x x}-\left(\frac{3}{2} u^{2}\left(u^{2}-u_{x}^{2}\right)+u^{3}\left(u-u_{x x}\right)\right)_{x}=0. \tag{1.8}$$

In [1], Anco and Recio discovered that the equation (1.2) has peakons. Equation (1.8) has the peakon solution

$$u(x, t)=c \varphi(x+c t)=c e^{-|x+c t|}, c \in \mathbb{R}. \tag{1.9}$$

Constantin and Strauss showed in [9] that peakons of a certain nonlinear dispersive equation are orbital stable. By using the method in [8], the orbital stability of peakons for other nonlinear wave equations was proved [5, 12, 17–19, 23]. More recently, Lu, Chen and Deng [20] have studied the peakon solutions of the gCH equation when $p=2$ and the orbital stability of the peakons. By using the dynamic system method, Lu, Lu and Chen [21] obtained some peakon and periodic peakon solutions to the modified Camassa-Holm equation, and Li [14] studied the dynamical behavior for the generalized Burger-Fisher equation and the Sharma-Tasso-Olver equation under different parametric conditions. In this paper, by the dynamic system method [10, 13, 15, 16] and the method in [9], we mainly study the case of $p=3$ in the gCH equation (1.2), which has higher degree of nonlinearity and integration than the former [20]. It is much more complex to construct the fifth degree polynomial to prove the stability of peakons, and we discover that the planar system has only one heteroclinic, which is different from the case of $p=2$ in [20].

Now, we state the main result of this paper.