Journal of Nonlinear Modeling and Analysis
Volume 4, Number 2, June 2022, 198–206
http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2022.198

Existence of Viscosity Solutions to a System of Hyperbolic Balance Laws

Qingning Zhang[1,†] and Min Cheng[1]

Abstract In this paper, the existence of viscous solutions of a hyperbolic equilibrium law system derived from the nonlinear entropy moment closure of a dynamic equation is established. In addition, by using the natural entropy of the system, some higher order estimates of some viscosity solutions are obtained.

Keywords System of hyperbolic balance laws, Viscosity solutions, The invariant region.

MSC(2010) 35L65, 35Q35.

1. Introduction

In this paper, we consider viscosity solutions to the following system of hyperbolic balance laws:

$$\begin{cases} \rho_t + \partial_x J = 0, \\ J_t + \partial_x(\rho\psi(\frac{J}{\rho})) = -J, \\ \rho(x,0) = \rho_0(x),\ J(x,0) = 0, \end{cases} \tag{1.1}$$

where ρ is the density and $u = \frac{J}{\rho}$ is the the velocity. ψ is given by:

$$\begin{aligned} \psi :\ & (-1,1) \to (0,+\infty) \\ & u \mapsto u^2 + \mathbb{G}'(\mathbb{G}^{-1}(u)) = \frac{\mathbb{F}''}{\mathbb{F}}(\mathbb{G}^{-1}(\frac{J}{\rho})). \end{aligned} \tag{1.2}$$

Here, $\mathbb{F}(\beta) = \frac{\sinh\beta}{\beta}$, $\mathbb{G}(\beta) = \coth\beta - \frac{1}{\beta} = \frac{\mathbb{F}'(\beta)}{\mathbb{F}(\beta)}$ and $\mathbb{G}$ is C^∞ diffeomorphism from $\mathbb{R}$ onto $(-1,1)$. From the definition, we know that $\mathbb{F}$, ψ are even functions, while $\mathbb{G}$ is an odd function, and ψ is strictly convex with

$$\mathbb{F}(0) = 1,\ \mathbb{G}(0) = 0,\ \psi(0) = \mathbb{G}'(0) = \frac{1}{3},\ \psi'(0) = 0, \tag{1.3}$$

$$\lim_{u\to\pm 1} \psi(u) = 1,\ \lim_{u\to\pm 1} \psi'(u) = \pm 2. \tag{1.4}$$

†the corresponding author.
Email address: zhangqingning@zjnu.edu.cn (Q. Zhang), mcheng@zjnu.edu.cn (M. Cheng)
[1]Department of Mathematics, Zhejiang Normal University, Jinhua, Zhejiang 321004, China

Direct calculation implies that the eigenvalues satisfy:

$$\lambda_i(u) = \frac{\psi'(u) \pm \sqrt{[\psi'(u)]^2 - 4u\psi'(u) + 4\psi(u)}}{2}, \ i = 1, 2, \tag{1.5}$$

with $\lambda_1(u) < u < \lambda_2(u)$ and $\lambda_i'(u) > 0$, and the corresponding eigenvectors are given by

$$r_i(u) = \begin{pmatrix} 1 \\ \lambda_i(u) \end{pmatrix}. \tag{1.6}$$

System (1.1) is strictly hyperbolic and genuinely nonlinear, and all the properties mentioned above are given in [1]. Moreover, it is shown that the corresponding homogeneous Riemann problem can be solved without smallness assumption. The existence of global weak solutions with vacuum for the isothermal Euler equations was proved in [4]. Diperna [2] gave the global weak solutions to the isentropic gas dynamics system with the vanishing viscosity method. For the Broadwell model, Lu [5] gave the existence of the viscosity solutions.

Now, we give the structure of the paper as follows: In Section 2, we review the existence theorem of invariant region. In Section 3, we prove the existence of invariant region and obtain the lower bound of the density. In Section 4, we apply entropy-entropy flux pairs to establish higher order estimation of viscosity solutions.

2. Preliminaries

In this section, we review the definition of invariant region and the theorem that we will apply to prove existence of invariant regions. Consider the following system:

$$\begin{cases} \partial_t v = \epsilon D(v,x)v_{xx} + M(v,x)v_x + f(v,t), & (x,t) \in \Omega \times \mathbb{R}_+, \\ v(0,x) = v_0(x), & x \in \Omega. \end{cases} \tag{2.1}$$

Here, $\epsilon > 0$, Ω is an open interval in $\mathbb{R}$, $D = D(v,x)$, and $M = M(v,x)$ are matrix-valued functions defined on an open subset $U \times V \subset \mathbb{R}^n \times \Omega$, $D \geq 0$. $v = (v_1, v_2 \dots v_n)$, and f is a smoothing mapping from $U \times \mathbb{R}_+$ into $\mathbb{R}^n$.

Definition 2.1. [6] A closed subset $\sum \subset \mathbb{R}^n$ is called a (positively) invariant region for the local solution defined by (2.1), if any solution $v(x,t)$ with its boundary and initial values in $\sum$ satisfies $v(x,t) \in \sum$, for all $x \in \Omega$ and $t \in [0,\sigma)$.

We consider the region $\sum$ of the form

$$\Sigma = \bigcap \{v \in V : G_i(v) \leq 0\}, \tag{2.2}$$

where G_i are smooth real-valued functions defined on an open subset of U, and for each i, the DG_i never vanishes.

Theorem 2.1. *[6] Let Σ be defined in* (2.2)*, and suppose that for all $t > 0$ and $v_0 \in \partial\Sigma$ ($G_i(v_0) = 0$ for some i), the following conditions hold:*

(1) DG_i at v_0 is a left eigenvector of $D(v_0,x)$ and $M(v_0,x)$, for all $x \in \mathbb{R}$;

(2) If $DG_iD(v_0,x) = \lambda DG_i$ with $\lambda \neq 0$, then G_i is strongly convex at v_0;