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Existence and Uniqueness of Solutions and Lyapunov-type Inequality for a Mixed Fractional Boundary Value Problem*

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Abstract In this paper, a Lyapunov-type inequality for a linear differential equation involving right Riemann-Liouville and left Caputo fractional derivatives under Sturm-Liouville boundary conditions is established. Furthermore, the existence of solutions for the corresponding nonlinear differential equation is obtained by fixed point theorems.

Keywords Fractional derivative, Lyapunov-type inequality, Fixed point theorem.

MSC(2010) 26A33, 34A40, 47H10.

1. Introduction

Fractional calculus is the theory of differential and integral of any order, which is the extension of integer order calculus. Since it can well describe the real world phenomena, it has been gaining popularity among scientists working on different subject. According to the actual needs, mathematicians give a variety of definitions of fractional derivatives and integrals [13]. The most commonly used fractional calculus operators are perhaps Riemann-Liouville and Caputo fractional integrals and derivatives.

Recently, Lyapunov-type inequalities for fractional differential equations have been widely used in various problems, including oscillation, disconjugacy and eigenvalue problems. This work was first done by Ferreira [4], who obtained a Lyapunov-type inequality for the following differential equations with Riemann-Liouville fractional derivative:

$$\begin{cases} D_{a+}^{\alpha}u(t)+q(t)u(t)=0, a<t<b, 1<\alpha<2,\\ u(a)=u(b)=0. \end{cases}$$

In 2018, Ntouyas et al., [16] summarized the development of Lyapunov inequalities in fractional differential equations. Moreover, many authors have obtained Lyapunov-type inequalities for mixed fractional differential equations [5,6,8,11,12].

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For example, Khaldi [12] considered the equation:

$$\begin{cases} -{}^C D_{b-}^{\alpha} D_{a+}^{\beta} u(t) + q(t)u(t) = 0, a < t < b, \\ u(a) = u(b) = 0, \end{cases}$$

where $0 < \beta \leq \alpha \leq 1$, $1 < \alpha + \beta \leq 2$.

At the same time, more and more attention has been paid to the existence of solutions for fractional boundary value problems [1–3,9,10,14,15,17,20]. In particular, fixed theorems have been extensively used to study the solutions of equations. For example, in [7], the authors investigated the existence of nonnegative solutions of fractional Liouville equation by using Krasnoselskill's fixed point theorem. In 2011, Samet et al., [19] introduced a new concept of $\alpha - \psi-$contractive type mappings and established fixed point theorems for such mappings in complete metric spaces. Motivated by [3, 12], in this paper, we consider the Lyapunov-type inequality for the following fractional differential equation:

$$\begin{cases} {}^C D_{a+}^{\beta} D_{b-}^{\alpha} u(t) + q(t)u(t) = 0, a < t < b, \\ u(b) = 0, pu(a) = \gamma D_{b-}^{\alpha} u(a), \end{cases} \tag{1.1}$$

where $0 < \alpha \leq \beta \leq 1$, $1 < \alpha + \beta \leq 2$, $p\gamma \leq 0$ and $p \neq 0$, ${}^C D_{a+}^{\beta}$ denotes the left Caputo derivative, D_{b-}^{α} denotes the right Riemann-Liouville derivative, u is the unknown function and $q \in C([a, b], \mathbb{R})$.

Furthermore, we obtain the existence of solutions for the corresponding nonlinear problem:

$$\begin{cases} {}^C D_{a+}^{\beta} D_{b-}^{\alpha} u(t) + f(t, \lambda u(t), I_{a+}^{\tau} u(t), I_{b-}^{\delta} u(t)) = 0, a < t < b, \\ u(b) = 0, pu(a) = \gamma D_{b-}^{\alpha} u(a), \end{cases} \tag{1.2}$$

where $\lambda, \tau, \delta > 0$ and $f \in C([a, b] \times \mathbb{R}^3, \mathbb{R})$.

This paper is organized as follows: In section 2, we introduce some basic concepts. In Section 3, we prove Lyapunov-type inequality and the existence of solutions. Finally, we give two examples to illustrate the theoretical results.

2. Preliminaries

In this section, we recall the basic concepts related to our work.

Definition 2.1 ([13,18])**.** The left and right Riemann-Liouville fractional integrals $I_{a+}^{\alpha} f$ and $I_{b-}^{\alpha} f$ of order $\alpha > 0$ are defined respectively by

$$(I_{a+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}},$$

$$(I_{b-}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)dt}{(t-x)^{1-\alpha}},$$

where Γ is the gamma function.

Definition 2.2 ([13,18])**.** The left and right Riemann-Liouville fractional derivatives $D_{a+}^{\alpha} f$ and $D_{b-}^{\alpha} f$ of order $\alpha > 0$ are defined respectively by

$$(D_{a+}^{\alpha} f)(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n (I_{a+}^{n-\alpha} f)(x),$$