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Some Properties of Solutions to the Novikov Equation with Weak Dissipation Terms

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Abstract In this paper, we investigate the Novikov equation with weak dissipation terms. First, we give the local well-posedness and the blow-up scenario. Then, we discuss the global existence of the solutions under certain conditions. After that, on condition that the compactly supported initial data keeps its sign, we prove the infinite propagation speed of our solutions, and establish the large time behavior. Finally, we also elaborate the persistence property of our solutions in weighted Sobolev space.

Keywords Blow-up scenario, Global existence, Large time behavior, Persistence property.

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1. Introduction

In this paper, we discuss the following Novikov equation with weak dissipation terms:

$$y_t + y_x u^2 + byuu_x + \lambda y = 0, \quad t > 0, x \in \mathbb{R}. \tag{1.1}$$

When $\lambda = 0$, it is a special case of the Holm-Staley b-family equations:

$$y_t + y_x u^k + byu^{k-1}u_x = 0, \quad t > 0, x \in \mathbb{R}, \tag{1.2}$$

where $k \geqslant 1$, $\lambda \in \mathbb{R}$, $u(x,t)$ denote the velocity field, $y(x,t) = u - u_{xx}$.

Holm and Staley [28] got the exchange of stability in the dynamics of solitary wave solutions under changes in the nonlinear balance, which was in a 1+1 evolutionary partial differential equation both related to shallow water waves and to turbulence.

When $k = 1$ and $b = 2$, equation (1.2) reduces to the famous Camassa-Holm equation [4], while, if $k = 1$ and $b = 3$, it reduces to the Degasperis-Procesi equation. These equations arise at various levels of approximation in shallow water theory, and possess a physics background with shallow water propagation, the bi-Hamiltonian structure, Lax pair and explicit solutions including classical soliton, cuspon and

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peakon solutions. Moreover, these two types of equations have been also extensively studied in [5, 10, 11, 13, 18, 19, 22, 26, 27, 39, 42, 53].

We know that Camassa-Holm equation is completely integrable. Definitely, the Camassa-Holm equation has many useful properties, for example, conservation rate, blow-up scenario Global existence and large time behavior for the support of the momentum density [30,40]. When it comes to the physical relevance of the Camassa-Holm and Degasperis-Procesi equation, we suggest the readers reading the book written by Constantin and Lannes [14]. In the $H^s, s > \frac{3}{2}$ space, the solution of the local well-posedness was proved in [11,36]. In [11,12,32,36,41], the blow up scenario was widely used. For the Camassa-Holm, the solution of Global existence and local solution was proved in [2, 3, 33]. They also proved orbital stability of the peak solution in [15], In [27], Himonas et al., gave the persistence and unique continuity of solution of Cassama-Holm equation. They discussed the large time behavior for the support of momentum density of the Camassa-Holm equation. They proved the limit of the support of momentum density as t goes to $+\infty$ in some sense. Moreover, the Degasperis-Procesi equation has been widely studied in [8, 9, 17, 31, 37, 44, 53].

When $k = 1$, for general b, the equation (1.2) was studied in [21, 54], which has established the local well-posedness and sufficient conditions on the initial data to guarantee the global existence of strong solutions in H^s, $s > \frac{3}{2}$. Blow-up scenario for equation (1.2) has been studied in [20, 43, 45, 47, 52], and some blow-up criteria was established in [16, 50]. Guan and Yin [24] studied the global existence and blow-up phenomenon of the integrable two-component Camassa-Holm shallow water system. Moreover, Liu and Yin [55] presented several conditions for the existence of global solutions. The large-time behavior of the supporting the momentum density for the Camassa-Holm equation was studied in [33]. [49] proposed a new method to show the persistence properties. Guo et al., [25] studied the large time behavior and persistence properties of solutions to the Camassa-Holm-type equation with higher-order nonlinearities. Here, we would like mention some related work of equation (1.2) in [7, 23, 29, 35, 38, 48, 51, 57].

In 2011, Zhu and Jiang [59] discussed the case of $k = 1$ in (1.2):

$$y_t + y_x u + byu_x + \lambda y = 0, \quad t > 0, x \in \mathbb{R}, \tag{1.3}$$

and got a new criterion on the blow-up phenomenon of the solution, the global existence and the persistence property of the solution. Zhang [56] considered the Camassa-Holm equation with weak dissipation terms. Niu and Zhang [45] established the local well-posedness of the inhomogeneous weak dissipation equation, which included both the weakly dissipative Camassa-Holm equation and the weakly dissipative Degasperis-Procesi equation as its special case. Zhou et al., [58] discussed the following more general equation:

$$y_t + y_x u^k + byu^{k-1}u_x + \lambda y = 0, \quad t > 0, x \in \mathbb{R}. \tag{1.4}$$

For equation (1.4), Zhou et al., [6, 34, 58] listed some existing conditions of global solution and some analytical properties of solution. When $k = 2$, the equation (1.4) would be the equation (1.1). The equation (1.1) could be rewritten as:

$$u_t + u^2 u_x + G * F(u) + \lambda u = 0, \tag{1.5}$$

where

$$F(u) = (6 - b)uu_x u_{xx} + 2u_x^3 + bu^2 u_x, \tag{1.6}$$