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Hopf Bifurcation Analysis of a Class of Abstract Delay Differential Equation*

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Abstract The dynamics of a class of abstract delay differential equations are investigated. We prove that a sequence of Hopf bifurcations occur at the origin equilibrium as the delay increases. By using the theory of normal form and centre manifold, the direction of Hopf bifurcations and the stability of the bifurcating periodic solutions is derived. Then, the existence of the global Hopf bifurcation of the system is discussed by applying the global Hopf bifurcation theorem of general functional differential equation.

Keywords Hopf bifurcation, Delay, Stability, Normal form, Periodic solution.

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1. Introduction

Since the last century, people have successively proposed a large number of delay differential equations problems in many fields of natural and social sciences such as galaxy evolution [9], optics [3, 20], nuclear physics [1, 31], chemical circulation systems, neural networks [33], population dynamics [28], ecosystems [24], infectious diseases [2, 17, 30], etc. For example, population growth model

$$N'(t) = K\left(1 - \frac{N(t-\tau)}{p}\right)N(t) \tag{1.1}$$

is a nonlinear delay differential equation. Neutral delay differential equation proposed in the study of energy loss in power networks

$$\dot{x}(t) = A\dot{x}(t-\tau) + Bx(t) - Cx(t-\tau) \tag{1.2}$$

is also a very typical example. The proposition of these problems has aroused increasing attention on the study of differential equations with delay. Similar arguments can be found in [6, 10, 11].

Before the 1930s, the research content of functional differential equations was limited to the special properties of some special types of equations. Volterra [25, 26]

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used the relationship between the functional differential range and some physical systems to define the energy function to observe the asymptotic behavior of system in a short time. This was a milestone in the development of the theoretical system of functional differential equations. With the in-depth study of these problems, the theory of functional differential equations has been continuously improved, and many monographs on the theory of functional differential equations have appeared. Bellman and Danskin [3], Bellman and Cooke [18] put forward the stability theorem of the linear difference differential equation of constant system, Krasovskii [12] also gave. Hale [7] explained the theory of functional differential equations more comprehensively from the aspects of stability, boundedness, periodic solutions, vibration and asymptotic properties, and almost periodic solutions. Further, Hale and Lunel published the Introduction to Functional Differential Equations in 1993, which made a good summary of the research on finite delay functional differential equations.

Delay differential equation is an important branch of differential equation. It is a kind of differential equation whose derivative function of time depends on the value of the solution at the past time point [22]. It is used to describe the motion phenomenon related to the state of motion and historical time. Specifically, the differential equation describing the development process of a specific system stated over time of the objective world is called a differential dynamic system. If the development of the system state depends not only on the current state, but also on the state of the system at certain moments or time segments in the past, this type of dynamic system is called a time-delay differential dynamic system [5].

Generally speaking, delay differential systems have more complex dynamic properties than corresponding ordinary differential systems. This is because the time lag can change the stability of the equilibrium point of the system and lead to the occurrence of Hopf branching and chaos. Therefore, it is a very meaningful subject to study the influence of time delay on the dynamics of the system. In fact, there is a wide literature on the dynamic systems with time delay, we refer the readers to [8, 16, 19, 21, 23, 32] respectively and references therein.

A two-agent opinion dynamical system with processing delay

$$\begin{cases} \dot{x}_1(t) = \frac{1}{2}\alpha a_{12}\left(x_2(t-\tau) - x_1(t-\tau)\right) \\ \dot{x}_2(t) = \frac{1}{2}\alpha a_{21}\left(x_1(t-\tau) - x_2(t-\tau)\right) \end{cases} \tag{1.3}$$

is discussed in [29]. The author transforms the dynamic problem into a kind of delay differential equation

$$\dot{x}(t) = \alpha p(x(t-\tau)), \tag{1.4}$$

and analyze the asymptotic stability of its origin. Wei [15] studied the dynamic properties of a scalar delay differential equation

$$\dot{x}(t) = -\gamma x(t) + \beta f(x(t-\tau)). \tag{1.5}$$

Equation (1.5) proved that the Hopf bifurcation sequence occurs with the increase of time delay at the equilibrium point. Furthermore, the results of the existence of the global Hopf bifurcation are studied, and the global existence of multiple periodic solutions is established.