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# On Two-point Boundary Value Problems for Second-order Difference Equation*

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**Abstract** In this paper, we aim to investigate the difference equation

$$\Delta^2 y(t-1) + |y(t)| = 0, \qquad t \in [1,T]_{\mathbb{Z}}$$

with different boundary conditions $y(0)=0$ or $\Delta y(0)=0$ and $y(T+1)=B$ or $\Delta y(T)=B$, where $T \geq 1$ is an integer and $B \in \mathbb{R}$. We will show that how the values of $T$ and $B$ influence the existence and uniqueness of the solutions to the about problem. In details, for the different problems, the $TB$-plane explicitly divided into different parts according to the number of solutions to the above problems. These parts of $TB$-plane for the value of $T$ and $B$ guarantee the uniqueness, the existence and the nonexistence of solutions respectively.

**Keywords** Second-order difference equation, Different boundary conditions, Boundary value problems.

**MSC(2010)** 39A27, 39A05.

## 1. Introduction

Let $c,d$ be two integers with $c<d$. We use $[c,d]_{\mathbb{Z}}$ to denote the set $\{c, c+1, \cdots, d\}$. Consider the following nonlinear second-order difference equation

$$\Delta^2 y(t-1) + |y(t)| = 0, \quad t \in [1,T]_{\mathbb{Z}} \tag{1.1}$$

with the boundary conditions (BCs)

$$y(0)=0,\ y(T+1)=B, \tag{1.2}$$

or

$$y(0)=0,\ \Delta y(T)=B, \tag{1.3}$$

or

$$\Delta y(0)=0,\ y(T+1)=B, \tag{1.4}$$

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or

$$\Delta y(0) = 0,\ \Delta y(T) = B, \tag{1.5}$$

where $T \in [1, \infty)_{\mathbb{Z}}$ and $B \in \mathbb{R}$, $\Delta$ is the forward difference operator satisfying $\Delta y(t) = y(t+1) - y(t)$ and $\Delta^2 y(t) = \Delta(\Delta y(t))$.

In the past few years, boundary value problems for difference equations have been widely studied in different disciplines such as computer science, economics, mechanical engineering, control systems (see [1,2,9]). Great efforts have been made to study the existence, multiplicity and uniqueness of solutions of boundary value problems with various boundary conditions (1.2)-(1.5) (see [3–8, 10–12] and the references therein). In [4], Bailey, Shampine and Waltman considered the nonlinear problem

$$\begin{cases} y'' + |y(t)| = 0,\ t \in [0, b], \\ y(0) = 0,\ y(b) = B, \end{cases} \tag{1.6}$$

and obtained the results: problem (1.6) has a unique solution for every number $B$, if $b < \pi$. However, if $b \geq \pi$, then there is one solution, no solution (existence fails), or more than one solutions (uniqueness fails), depending upon the number $B$.

Motivated by the above facts, it is natural to raise a question: if the interval length is larger than a certain value, whether or not to influence the existence and uniqueness of the solution for the difference equation (1.1) with BCs (1.2), (1.3), (1.4), (1.5) respectively? In this paper, we will show how the values of $T$ and $B$ influence the existence and uniqueness of the solutions to the problem (1.1) with BCs (1.2), (1.3), (1.4), (1.5) respectively.

## 2. Preliminaries

Let $y(t)$ be a solution of (1.1). If $y(t) \geq 0$, then it is a solution of the equation

$$\Delta^2 y(t-1) + y(t) = 0, \qquad t \in [1, T]_{\mathbb{Z}}. \tag{2.1}$$

If $y(t) \leq 0$, it is a solution of the equation

$$\Delta^2 y(t-1) - y(t) = 0, \qquad t \in [1, T]_{\mathbb{Z}}. \tag{2.2}$$

Let us consider equation (2.1) and equation (2.2) with the initial conditions $y(0) = 0$ and $\Delta y(0) = 0$ respectively.

(a) We rewrite (2.1) as

$$y(t+1) - y(t) + y(t-1) = 0, \tag{2.3}$$

and the general solution of the difference equation of (2.1) is

$$y(t) = c_1 \cos \frac{\pi}{3} t + c_2 \sin \frac{\pi}{3} t, \tag{2.4}$$

where $c_1,\ c_2$ are arbitrary constants. Therefore, the solution of (2.1) which satisfies $y(0) = 0$ is as follows:

$$y(t) = c \sin \frac{\pi}{3} t, \tag{2.5}$$