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Finite Difference Method for Inhomogeneous Fractional Dirichlet Problem

Jing Sun, Weihua Deng* and Daxin Nie

School of Mathematics and Statistics, Gansu Key Laboratory of Applied Mathematics and Complex Systems, Lanzhou University, Lanzhou 730000, P.R. China

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Abstract. We make the split of the integral fractional Laplacian as

$$(-\Delta)^s u = (-\Delta)(-\Delta)^{s-1} u,$$

where $s \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. Based on this splitting, we respectively discretize the one- and two-dimensional integral fractional Laplacian with the inhomogeneous Dirichlet boundary condition and give the corresponding truncation errors with the help of the interpolation estimate. Moreover, the suitable corrections are proposed to guarantee the convergence in solving the inhomogeneous fractional Dirichlet problem and an $\mathcal{O}(h^{1+\alpha-2s})$ convergence rate is obtained when the solution $u \in C^{1,\alpha}(\bar{\Omega}_n^\delta)$, where n is the dimension of the space, $\alpha \in (\max(0, 2s-1), 1]$, δ is a fixed positive constant, and h denotes mesh size. Finally, the performed numerical experiments confirm the theoretical results.

AMS subject classifications: 65N06, 35R11, 65N15

Key words: One- and two-dimensional integral fractional Laplacian, Lagrange interpolation, operator splitting, finite difference, the inhomogeneous fractional Dirichlet problem, error estimates.

1. Introduction

Fractional Laplacian is of wide interest to both pure and applied mathematicians, and also has extensive applications in physical and engineering community [7, 19]. Based on the splitting of the integral fractional Laplacian, we provide the finite difference approximations for the one- and two-dimensional cases of the operator. Then the

*Corresponding author. *Email addresses:* `js@lzu.edu.cn` (J. Sun), `dengwh@lzu.edu.cn` (W. Deng), `ndx1993@163.com` (D. Nie)

approximations are used to numerically solve the inhomogeneous fractional Dirichlet problem, i.e.,

$$\begin{cases}(-\Delta)^s u(\mathbf{x}) = f(\mathbf{x}) & \text{in } \Omega_n,\\ u(\mathbf{x}) = g(\mathbf{x}) & \text{in } \Omega_n^c,\end{cases} \tag{1.1}$$

where $\Omega_n \subset \mathbb{R}^n$ $(n = 1, 2)$ is a bounded domain and $\Omega_n^c = \mathbb{R}^n \backslash \Omega_n$ denotes the complement of Ω_n; $g(\mathbf{x}) = 0$ in Ω_n, $g(\mathbf{x}) \in L^\infty(\mathbb{R}^n)$, and **supp** $g(\mathbf{x})$ is bounded; $(-\Delta)^s u(\mathbf{x})$ is the integral fractional Laplacian, which can be defined by [1, 7]

$$(-\Delta)^s u(\mathbf{x}) = c_{n,s}\text{P.V.}\int_{\mathbb{R}^n} \frac{u(\mathbf{x}) - u(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^{n+2s}} d\mathbf{y} \tag{1.2}$$

with

$$c_{n,s} = \frac{2^{2s} s\Gamma(\frac{n}{2} + s)}{\pi^{\frac{n}{2}}\Gamma(1 - s)},$$

and $s \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. And the Fourier transform of $(-\Delta)^s u(\mathbf{x})$ can be written as [1, 7]

$$\mathcal{F}\big((-\Delta)^s u(\mathbf{x})\big)(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{2s}\mathcal{F}(u), \tag{1.3}$$

where $\mathcal{F}$ stands for the Fourier transform.

Lévy process is one of the most commonly used models for describing anomalous diffusion phenomena [3, 22], especially α-stable Lévy process. Fractional Laplacian is introduced as the infinitesimal generator of α-stable Lévy process [7, 12]. Since the singularity and non-locality, numerical approximation of fractional Laplacian is still a challenging topic. In the past few decades, finite difference method has been widely used to approximate fractional derivatives [2, 6, 9, 10, 12–18, 21, 23, 25]. Among them, [15–18] discretize time fractional Caputo derivative by L_1 method and convolution quadrature method; [6, 23] provide weighted and shifted Grünwald difference method to discretize fractional Riesz derivative; as for fractional Laplacian, [9, 10, 12, 13] propose the finite difference scheme for solving d-dimensional $(d = 1, 2, 3)$ fractional Laplace equation with homogeneous Dirichlet boundary condition; moreover, the finite difference schemes provided in [21, 25] for tempered fractional Laplacian with $\lambda = 0$ still apply to fractional Laplacian.

Different from the previous finite difference scheme for fractional Laplacian, we split it into the product of $(-\Delta)$ and $(-\Delta)^{s-1}$ according to its Fourier transform form, where $-\Delta$ denotes the classical Laplace operator, and $(-\Delta)^{s-1}$ (the exponent $s-1 < 0$) is a non-local operator without hyper-singularity (for the detailed definition, see (2.3)). Then we use the Lagrange interpolation to discretize $(-\Delta)^{s-1}$ and the finite difference to $-\Delta$ for one- and two-dimensional cases, respectively. Moreover, some corrections are made to ensure the convergence when using our discretization to solve Eq. (1.1). Compared with the discretizations in [9, 10], our scheme can deal with the inhomogeneous fractional Dirichlet problem more easily and accurately. Different from the discretizations proposed in [21, 25], the current discretization can produce a Toeplitz matrix in one-dimensional case and a block-Toeplitz-Toeplitz-block for two-dimensional case; so