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A Note on Obata Equations on Manifolds with Boundary

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Abstract. We analyze complete manifolds with boundary which admit solutions to Obata type equations $\nabla^2 f - fg = 0$ and $\nabla^2 f = g$ under various boundary conditions. Given the existence of interior critical points, such manifolds are domains in the hyperbolic space and the Euclidean space respectively.

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1 Introduction

Obata's theorem [7] asserts that a complete Riemannian manifold (M,g) is isometric to the standard sphere if and only if there exists a non-trivial function f satisfying

$$\nabla^2 f + fg = 0. \tag{1.1}$$

Following the proof of Obata's theorem, one can easy prove the following hyperbolic and Euclidean analogs:

Theorem 1.1 (Theorem A). *Let (M,g) be a complete manifold. Suppose f is a positive solution of*

$$\nabla^2 f - fg = 0, \tag{1.2}$$

with critical point. Then M is isometric to the standard hyperbolic space $\mathbb{H}^n$.

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Theorem 1.2 (Theorem B)**.** *Let (M,g) be a complete manifold. Suppose f is a solution of*

$$\nabla^2 f - g = 0. \tag{1.3}$$

Then M is isometric to the standard Euclidean space $\mathbb{R}^n$.

Obata's theorem characterizes the equality case of Lichnerowicz's first eigenvalue estimates for the Laplacian on closed manifolds with $Ric \geq n-1$. Later, Reilly [8] obtained first eigenvalue estimates for the Laplacian on manifolds with boundary under the Dirichlet boundary condition. Eigenvalue estimates with Neumann boundary condition were studied independently by Escobar [4] and Xia [11]. The characterization of equality cases amounts to the study of the Obata equation on manifolds with boundary under the Dirichlet and Neumann boundary conditions respectively. Recently, Chen, Lai and Wang [2] carefully discussed the Obata equation with Robin boundary condition.

In this note, we analyze complete manifolds with boundary which admit solutions to (1.2) or (1.3) under various boundary conditions. In order to yield results of geometric significance, it is necessary to pose a primary assumption: the existence of interior critical point. Without such assumption, M only has a warped product structure and the analysis is less interesting. The following are our main theorems.

Theorem 1.3. *Let (M,g) be a complete, connected manifold with boundary. Suppose f is a positive solution of $\nabla^2 f - fg = 0$ with interior critical point. Then we have the following:*

1. *If $f = c$ on ∂M, then M is isometric to a geodesic ball in $\mathbb{H}^n(-1)$.*
2. *If $\frac{\partial f}{\partial \nu} = a (a > 0)$ on ∂M then M is isometric to one of the following*
   - *a geodesic ball of radius $\sinh^{-1}(a)$ in $\mathbb{H}^n(-1)$;*
   - *an unbounded domain in $\mathbb{H}^n(-1)$ and each boundary component is totally geodesic.*
3. *If $\frac{\partial f}{\partial \nu} = af (0 < a < 1)$ on ∂M then M is isometric to a domain in $\mathbb{H}^n(-1)$ bounded by $S^k(\sqrt{\frac{a^2}{1-a^2}}) \times \mathbb{H}^{n-k-1}(a^2-1)$, for some $k = 0,1,\cdots,n-1$.*

Theorem 1.4. *Let (M,g) be a complete, connected manifold with boundary. Suppose f is a solution of $\nabla^2 f = g$ with interior critical point. Then we have the following:*

1. *If $f = c$ on ∂M, then M is isometric to a Euclidean ball.*
2. *If $\frac{\partial f}{\partial \nu} = a$ $(a > 0)$ on ∂M then M is isometric to a domain in $\mathbb{R}^n$ bounded by $S^k(a) \times \mathbb{R}^{n-k-1}$, for some $k = 0,1,\cdots n-1$.*
3. *If $\frac{\partial f}{\partial \nu} = af$ $(a > 0)$ on ∂M, then M is isometric to a Euclidean ball of radius $\frac{1}{a}$.*

Remark 1.1. Our notation conventions are as follows:

- $S^k(\sqrt{\frac{a^2}{1-a^2}})$: the round sphere of radius $\sqrt{\frac{a^2}{1-a^2}}$.