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A New Method for Computing the Expected Hitting Time between Arbitrary Different Configurations of the Multiple–Urn Ehrenfest Model

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Abstract. We study a multiple–urn version of the Ehrenfest model. In this setting, we denote the n urns by Urn 1 to Urn n, where $n \geq 2$. Initially, M balls are randomly placed in the n urns. At each subsequent step, a ball is selected and put into the other $n-1$ urns with equal probability. The expected hitting time leading to a change of the M balls' status is computed using the method of stopping times. As a corollary, we obtain the expected hitting time of moving all the M balls from Urn 1 to Urn 2.

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1 Introduction

We extend the classical two–urn Ehrenfest model to the multiple–urn case. Label the n urns by Urn 1 to Urn n, where $n \geq 2$. At the beginning, M balls are arbitrarily placed in the n urns. Then at each time, one ball is chosen at random, taken from the current urn it resides in, and placed in one of the other $n-1$ urns with equal probability. This model can be treated as a symmetric simple random walk on the graph $G_M = (V_M, E_M)$, where $V_M = \{1, \cdots, n\}^M$, and E contains edges connecting two vertices in V_M if exactly one of their components differs. Here the subscript "M" is to stress that the number of balls is M. Therefore, G_M is a transitive graph (that is, for any $e, e' \in E_M$, there is an automorphism of the graph that takes e to e') with n^M vertices, and each vertex has common degree $(n-1)M$. Strictly speaking, if we let $X_t = (X_t^{(1)}, \cdots, X_t^{(M)})$ be the state at time $t = 0, 1, \cdots$, where $X_t^{(i)}$ is the number of the urn in which the ith ball resides at

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time t, then $\{X_t:\ t=0,1,\cdots\}$ is a time homogeneous Markov chain on V_M with transition probability

$$p_{(x_1,\cdots,x_M),(y_1,\cdots,y_M)}=\begin{cases}\frac{1}{(n-1)M}, & \text{if there exists } i \text{ s.t. } x_i\neq y_i, \text{ and } x_j=y_j \text{ for } j\neq i;\\ 0, & \text{otherwise.}\end{cases} \tag{1.1}$$

For $x_1,\cdots,x_M\in\{1,2,\cdots,n\}$, denote by

$$T_{(x_1,\cdots,x_M)}=\inf\{t\geq 0:\ X_t=(x_1,\cdots,x_M)\}$$

the first time that $\{X_t\}$ hits state $(x_1,\cdots,x_M)$. Our main result is described in the following theorem.

Theorem 1.1. *For any two different configurations $(a_1,\cdots,a_M),(b_1,\cdots,b_M)\in\{1,\cdots,n\}^M$, denote $L=\sum_{i=1}^M \mathbf{1}_{\{a_i=b_i\}}$. Then*

$$\mathbb{E}(T_{(b_1,\cdots,b_M)}\mid X_0=(a_1,\cdots,a_M))=\sum_{k=L}^{M-1}\frac{(n-1)^{k+1}}{\binom{M-1}{k}}\sum_{i=0}^{k}\frac{\binom{M}{i}}{(n-1)^i},$$

where $\mathbf{1}_{\{\cdot\}}$ denotes the indicator function, and $\binom{n}{m}:=\dfrac{n!}{m!(n-m)!}$ $(0\leq m\leq n)$ denotes the combinatorial number.

Note that when $L=M-1$, the righthand side of Theorem 1.1 becomes

$$(n-1)^M\sum_{i=0}^{M-1}\frac{\binom{M}{i}}{(n-1)^i},$$

which equals to n^M-1. This is a well known result in Markov chains.

As a special case, we obtain the following corollary, which provides the expected hitting time of moving all balls from Urn 1 to Urn 2.

Corollary 1.2. We have

$$\mathbb{E}\left(T_{\underbrace{(2,2,\cdots,2)}_{M}}\ \middle|\ X_0=\underbrace{(1,1,\cdots,1)}_{M}\right)=\frac{(n-1)M}{n}\sum_{k=1}^{M}\frac{n^k}{k}.$$

Remark 1.1. (1) Chen *et al.* [3] proved Corollary 1.2 for the special case $n=3$ by using the method of electric networks. They conjectured that the result for general multiple–urn case should be of the form as stated in Corollary 1.2.

(2) Corollary 1.2 is a special case of Theorem 1.1 by letting $L=0$. This is not a straightforward result. A key step is to establish the equality

$$\sum_{k=0}^{M-1}\frac{(n-1)^k}{M\binom{M-1}{k}}\sum_{i=0}^{k}\frac{\binom{M}{i}}{(n-1)^i}=\frac{1}{n}\sum_{k=1}^{M}\frac{n^k}{k}. \tag{1.2}$$