J. Math. Study
doi: 10.4208/jms.v55n3.22.04

Vol. **55**, No. 3, pp. 271-280
September 2022

Optimal System and Invariant Solutions of the Hyperbolic Geometric Flow

Han Zhang[1] and Zenggui Wang[1,*]

[1] *School of Mathemtics Science, Liaocheng University, Liaocheng 252000, China;*

Received February 20, 2021; Accepted April 19, 2022;
Published online August 31, 2022.

Abstract. By analyzing Lie symmetric algebra of the hyperbolic geometric flow, the one-dimensional optimal system of the symmetries to the equation is obtained, and we use similarity reduction to find the reduced equation. By solving the reduced equations, the invariant solutions of the hyperbolic geometric flow are finally obtained.

AMS subject classifications: 22E46, 53C35, 57S20

Key words: Lie symmetry analysis, hyperbolic geometric flow, one-dimensional optimal system, invariant solutions.

1 Introduction

With the development of the times, partial differential equations (PDEs) are widely used in geometry. Based on the measured wave characteristics and Ricci flow, Kong *et al.* proposed a new geometric flow–hyperbolic geometric flow on Riemann surface [1-3]. Because this hyperbolic geometric flow combines partial differential equations, physics and differential geometry, so it is essentially different from Ricci flow. At the same time, Kong et al have also studied some special types of geometric flows, such as hyperbolic mean flow [4-5] and so on.

Let (M,g) be an n-dimensional complete Riemann flow, firstly, Kong *et al.* proposed the form of standard hyperbolic geometric flow in [6],

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -2R_{ij}, \tag{1.1}$$

where g_{ij} is defined as a family of Riemannian measures on M. Then under the condition of keeping the volume of flow unchanged, Kong *et al.* derived the normalized form of

*Corresponding author. *Email addresses:* `zh19960604@163.com` (Zhang H), `wangzenggui@lcu.edu.cn` (Wang Z)

the hyperbolic geometric flow. Finally, based on the work of normalization, Kong *et al.* defined the general form of hyperbolic geometric flow in [6]:

$$\frac{\partial^2 g_{ij}}{\partial t^2}+2R_{ij}+\mathcal{F}_{ij}(g,\frac{\partial g}{\partial t})=0, \tag{1.2}$$

where $\mathcal{F}_{ij}$ is not only a smooth function, but also related to the Riemann metric g and $\frac{\partial g}{\partial t}$. Because of the different forms of $\mathcal{F}_{ij}$, Kong *et al.* divided Eq. (1.2) into three cases, namely standard hyperbolic geometric flow, Einstein hyperbolic geometric flow and dissipative hyperbolic geometric flow. In [7], some properties of the short-time existence and nonlinear stability of hyperbolic geometric flow are given by Dai et al. In [8], by estimating some solutions of linear wave equations with two spatial variables, Kong *et al.* gave the lower lifetime bounds for classical solutions of hyperbolic geometric flows with asymptotically flat initial Riemannian surfaces. In [9], firstly, Dai *et al.* gave some interesting exact solutions to the dissipative hyperbolic geometric flow, then introduced a new concept: the hyperbolic Ricci soliton, and described some of its geometric properties, finally, the existence and uniqueness theorem of the dissipative hyperbolic geometric flow is established.

The purpose of this paper is to find the symmetries of the hyperbolic geometric flow by using the classical Lie group method [10-11], and then we investigate one-dimensional optimal system of the symmetries to the equation by using the method in [12-15], finally, by solving reduced equations, we obtain invariant solutions of the hyperbolic geometric flow.

In the end, this paper is arranged as follows. In Section 2, we mainly concentrate on acquiring one-dimensional optimal system of the symmetries to the hyperbolic geometric flow. In Section 3, we construct reduced equations and invariant solutions. In Section 4, another symmetry of the hyperbolic geometric flow is discussed. Some conclusions are shown in the last part.

2 One-dimensional optimal system of the symmetries to the hyperbolic geometric flow

2.1 Lie symmetry analysis of hyperbolic geometric flow

In fact, we all know that any Riemannian surface can at least be conformal to a flat surface locally, that is to say

$$g_{ij}=u(t,x,y)\delta_{ij}, \tag{2.1}$$

in which δ_{ij} stands for Kronecker symbol, $u(t,x,y)$ is a smooth function and satisfies $u(t,x,y)>0$. In this case, the scalar curvature R on the Riemannian surface is given by

$$R=-\frac{\ln u}{u}. \tag{2.2}$$