J. Math. Study
doi: 10.4208/jms.v56n1.23.04

Vol. **56**, No. 1, pp. 103-110
March 2023

Group-theoretical Property of Slightly Degenerate Fusion Categories of Certain Frobenius-Perron Dimensions

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Received February 22, 2022; Accepted July 22, 2022;
Published online October 11, 2022.

Abstract. Let p,q be odd primes, and let d be an odd square-free integer such that $(pq,d)=1$. We show that slightly degenerate fusion categories of Frobenius-Perron dimensions $2p^2q^2d$, $2p^2q^3d$ and $2p^3q^3d$ are group-theoretical.

AMS subject classifications: 18M20

Key words: Group-theoretical fusion category, slightly degenerate fusion category.

1 Introduction

Throughout this paper, we work over an algebraically closed field $\mathbb{K}$ of characteristic zero, $\mathbb{K}^*:=\mathbb{K}\backslash\{0\}$, $\mathbb{Z}_r:=\mathbb{Z}/r$, $r\in\mathbb{N}$. For any semisimple $\mathbb{K}$-linear finite abelian category $\mathcal{C}$, we use $\mathcal{O}(\mathcal{C})$ to denote the set of isomorphism classes of simple objects of $\mathcal{C}$.

Recall that a braided fusion category $\mathcal{C}$ is said to be slightly degenerate, if its Müger center $\mathcal{C}'$ is braided equivalent to sVec, the category of finite-dimensional super vector spaces over $\mathbb{K}$; while $\mathcal{C}$ is a non-degenerate fusion category if $\mathcal{C}'\cong$ Vec, the category of finite-dimensional vector spaces over $\mathbb{K}$. In [9], the first named author studied slightly degenerate fusion categories of various Frobenius-Perron dimensions. In particular, for odd primes p and q, slightly degenerate fusion categories of Frobenius-Perron dimension $2p^mq^nd$ are always integral and solvable [9, Corollary 3.4, Proposition 3.14], where d is an odd square-free integer such that $(pq,d)=1$. Moreover, non-degenerate fusion categories $\mathcal{C}$ of Frobenius-Perron dimensions p^2q^3d and p^3q^3d are group-theoretical [10], that is, $\mathcal{Z}(\mathcal{C})$ are braided equivalent to Drinfeld centers of certain pointed fusion categories.

Recently, for odd primes p and q, let d be an odd square-free integer such that $(pq,d)=1$, it was proved that a slightly degenerate fusion category $\mathcal{C}$ of Frobenius-Perron dimension $2p^mq^nd$ always contains a non-degenerate fusion subcategory $\mathcal{C}(\mathbb{Z}_d,\eta)$ [8, Corollary

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4.7], where $\mathcal{C}(\mathbb{Z}_d,\eta)$ is a non-degenerate fusion category determined by the metric group $(\mathbb{Z}_d,\eta)$ and $\eta:\mathbb{Z}_d\to\mathbb{K}^*$ is a non-degenerate quadratic form. Thus, $\mathcal{C}\cong\mathcal{C}(\mathbb{Z}_d,\eta)\boxtimes\mathcal{D}$ as braided fusion category by [2, Theorem 3.13], where $\mathcal{D}$ is a slightly degenerate fusion category of Frobenius-Perron dimensions $2p^mq^n$; in particular, it is easy to see that if $m\leq 1$ or $n\leq 1$ then $\mathcal{C}$ is nilpotent and group-theoretical by [1, Theorem 6.10, Corollary 6.8]. In this paper, we continue to study the structures of slightly degenerate fusion categories of FP-dimensions $2p^2q^2$, $2p^2q^3$ and $2p^3q^3$, we prove that they are also group-theoretical fusion categories, see Theorems 3.1, 3.2 and 3.3.

The paper is organized as follows. In Section 2, we recall some basic properties of fusion categories and braided fusion categories that we use throughout. In Section 3, we prove our main theorems: Theorems 3.1, 3.2 and 3.3, respectively.

2 Preliminaries

In this section, we will recall some most used results about fusion categories and braided fusion categories, we refer the readers to [1–5].

Let G be a finite group. A fusion category $\mathcal{C}=\oplus_{g\in G}\mathcal{C}_g$ is a G-graded fusion category, if for any $g,h\in G$, $\mathcal{C}_g$ is a $\mathbb{K}$-linear full abelian subcategories, $\mathcal{C}_g\otimes\mathcal{C}_h\subseteq\mathcal{C}_{gh}$ and $(\mathcal{C}_g)^*\subseteq\mathcal{C}_{g^{-1}}$. A G-grading of fusion category $\mathcal{C}=\oplus_{g\in G}\mathcal{C}_g$ is faithful if $\mathcal{C}_g\neq 0$ for any $g\in G$. Note that the trivial component $\mathcal{C}_e$ is a fusion subcategory of $\mathcal{C}$, so when the G-grading $\mathcal{C}=\oplus_{g\in G}\mathcal{C}_g$ is faithful, $\mathcal{C}$ is also called a G-extension of $\mathcal{C}_e$.

Let $\mathcal{C}_{\rm ad}$ be the adjoint fusion subcategory of $\mathcal{C}$, that is, $\mathcal{C}_{\rm ad}$ is generated by simple objects Y such that $Y\subseteq X\otimes X^*$ for some simple object X of $\mathcal{C}$, then $\mathcal{C}$ has a faithful $U_{\mathcal{C}}$-grading with $\mathcal{C}_{\rm ad}$ be the trivial component [5, Theorem 3.5], $U_{\mathcal{C}}$ is called the universal grading group of $\mathcal{C}$. Moreover, for any other faithful G-grading of $\mathcal{C}=\oplus_{g\in G}\mathcal{C}_g$, there exists a surjective group homomorphism $\phi:U_{\mathcal{C}}\to G$ by [5, Corollary 3.7].

Given a fusion category $\mathcal{C}$, there exists a unique ring homomorphism FPdim(-) from the Grothendieck ring $Gr(\mathcal{C})$ to $\mathbb{K}$ [3, Theorem 8.6], which satisfies that $\mathrm{FPdim}(X)\geq 1$ is an algebraic integer for any object $X\in\mathcal{O}(\mathcal{C})$. $\mathrm{FPdim}(X)$ is called the Frobenius-Perron dimension of the object X and the Frobenius-Perron dimension of fusion category $\mathcal{C}$ is defined by the following sum

$$\mathrm{FPdim}(\mathcal{C}):=\sum_{X\in\mathcal{O}(\mathcal{C})}\mathrm{FPdim}(X)^2.$$

A simple object $X\in\mathcal{C}$ is invertible if $X\otimes X^*=X^*\otimes X=I$, the unit object, equivalently $\mathrm{FPdim}(X)=1$. Recall that a fusion category $\mathcal{C}$ is pointed if all simple objects of $\mathcal{C}$ are invertible. Let $\mathcal{C}$ be a pointed fusion category, then $\mathcal{C}$ is tensor equivalent to the fusion category Vec_G^ω of G-graded finite-dimensional vector spaces over $\mathbb{K}$, where $\mathcal{O}(\mathcal{C})\cong G$ is a finite group, $\omega\in Z^3(G,\mathbb{K}^*)$ is a 3-cocycle. Given an arbitrary fusion category $\mathcal{C}$, we denote by $\mathcal{C}_{\rm pt}$ the maximal pointed fusion subcategory of $\mathcal{C}$ below, that is, $\mathcal{C}_{\rm pt}$ is the fusion subcategory generated by invertible objects of $\mathcal{C}$, and $G(\mathcal{C}):=\mathcal{O}(\mathcal{C}_{\rm pt})$.