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The Eigenvalues of a Class of Elliptic Differential Operators

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Abstract. Consider (M,g) as an n-dimensional compact Riemannian manifold. In this paper we are going to study a class of elliptic differential operators which appears naturally in the study of hypersurfaces with constant mean curvature and also the study of variation theory for 1-area functional.

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1 Introduction

Finding bounds for the eigenvalue of the Laplacian operator on a given manifold is a key aspects in Riemannian geometry. In the recent years, because of the theory of self-adjoint operators, the spectral properties of linear Laplacian studied extensively. Most of the mathematicians are generally interested in the spectrum of the Laplacian on compact manifolds with or without boundary or noncompact complete manifolds. Because in these cases, the linear Laplacian can be uniquely extended to self-adjoint operators (see [1,2]). For these purposes, mathematicians study the various extensions of such operators which are found in different theories in Riemannian and differential geometry.

As a first extension, consider M as a complete manifold. Let $f:M\longrightarrow\mathbb{R}$ be a smooth function on M or $f\in W^{1,p}(M)$ where $W^{1,p}(M)$ is the Sobolev space. The p-Laplacian of f for $1<p<\infty$ is defined as

$$\Delta_p f=\operatorname{div}\left(|\nabla f|^{p-2}\nabla f\right)=|\nabla f|^{p-2}\Delta f+(p-2)|\nabla f|^{p-4}(\operatorname{Hess} f)(\nabla f,\nabla f),$$

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where

$$(\mathrm{Hess} f)(X,Y) = \nabla(\nabla f)(X,Y) = X(Yf) - (\nabla_X Y)f, \qquad X,Y \in \chi(M).$$

The first eigenvalues of p-Laplace operator in both Dirichlet and Neumann cases have been studied in many papers (see [3–5]). If $\mu_{1,p}$ denotes the first Neumann eigenvalue of p-Laplace operator then as an example it was proved before in [3] that

Proposition 1.1 (Matei). *Let M be an n-dimensional complete Riemannian manifold and let K be a real constant such that $Ric^M \geq (n-1)K$. Then for any $x_0 \in M$, $r_0 \in (0,d_M)$ and $p \geq 2$,*

$$\mu_{1,p}(B(x_0,r_0)) \leq \mu_{1,p}(V_n(K,r_0)),$$

where $B(x_0,r_0)$ is geodesic ball in M centered at x_0 and radius r_0, $V_n(K,r_0)$ is geodesic ball with radius r_0 in model space, i.e. the n-dimensional simply connected space form with constant sectional curvature K. Also Ric^M and d_M denote the Ricci curvature tensor in M and the diameter of M respectively.

The other well known extension of Laplace operator is the weighted Laplace operator which is defined as $\Delta_f = \Delta - \nabla f \cdot \nabla$ and it acts as the Laplace operator in weighted manifolds, i.e. manifolds with density $e^{-f}dv$ (see [6,7]).

Another extension of Laplace operator is the elliptic divergence type operator

$$L_T f = \mathrm{div}(T\nabla f),$$

where T is a positive definite symmetric (1,1)-tensor field on a complete Riemannian manifold M. This operator is studied in [8] by second author before. Also the general case of this operator is defined as

$$L_{T,\eta} f = \mathrm{div}(T\nabla f) - \langle \nabla \eta, T(\nabla f)\rangle,$$

where $\eta \in C^2(M)$ and the eigenvalue problem for this operator is studied in [9]. As it was mentioned before, the first eigenvalue of these operators on a compact manifold M has been studied extensively in recent mathematical publications. Many connections between these invariants and other geometrical invariants have lead to some results of i-th eigenvalue of these operators. As an example in [9].

Proposition 1.2 (Gomez and Miranda). *Let Ω be a domain in an n-dimensional complete Riemannian manifold M isometrically immersed in $\mathbb{R}^m$, λ_i be the i-th eigenvalue of $L_{T,\eta}$ and f_t its corresponding normalized real-valued eigenfunction. Then*

$$\begin{aligned}&\mathrm{tr}(T)\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\\ &\leq \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\Big((m-n)^2A_0^2T_*^2 + (T_0+T_*\eta_0) + 4(T_0+T_*\eta_0)||T(\nabla f_i)||_{L^2} + 4\lambda_i\Big),\end{aligned}$$

where $A_0 = \max\{\sup_\Omega |A_{e_k}|, k = n+1,\ldots,m\}$, A_{e_k} is the Weingarten operator of the immersion with respect to e_k, $\eta_0 = \sup_{\bar\Omega}|\nabla\eta|$, $T_* = \sup_{\bar\Omega}|T|$ and $T_0 = \sup_{\bar\Omega}|\mathrm{tr}(\nabla T)|$.