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Convergence of Physics-Informed Neural Networks Applied to Linear Second-Order Elliptic Interface Problems

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Abstract. With the remarkable empirical success of neural networks across diverse scientific disciplines, rigorous error and convergence analysis are also being developed and enriched. However, there has been little theoretical work focusing on neural networks in solving interface problems. In this paper, we perform a convergence analysis of physics-informed neural networks (PINNs) for solving second-order elliptic interface problems. Specifically, we consider PINNs with domain decomposition technologies and introduce gradient-enhanced strategies on the interfaces to deal with boundary and interface jump conditions. It is shown that the neural network sequence obtained by minimizing a Lipschitz regularized loss function converges to the unique solution to the interface problem in H^2 as the number of samples increases. Numerical experiments are provided to demonstrate our theoretical analysis.

AMS subject classifications: 65N12, 82B24, 68T07

Key words: Elliptic interface problems, generalization errors, convergence analysis, neural networks.

1 Introduction

Deep learning in the form of deep neural networks (DNNs) has been effectively used in diverse scientific disciplines beyond its traditional applications. In particular, thanks to their potential nonlinear approximation power [1–3], DNNs are being exploited to construct alternative approaches for solving partial differential equations (PDEs), e.g., the

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deep Ritz method (DRM) [4] and physics-informed neural networks (PINNs) [5]. The key idea of these methods is to reformulate the solution to a PDE with a closed-form expression in the form of a neural network, the parameters of which are obtained by minimizing a physics-informed loss given by the corresponding PDE. The original works on the use of neural networks to solve PDEs were proposed in the 1990*s* [6,7], and this idea has recently been revisited with the renaissance of neural networks and the development of deep learning techniques; see e.g., [8–12] and references therein.

Elliptic interface problems are a widespread class of problems in scientific computing with many applications across diverse fields; see e.g. [13–16]. There are many accurate and efficient numerical methods in the literature for interface problems, such as the finite element method (FEM) [17,18], the discontinuous Galerkin method (DG) [19,20], the immersed interface method (IIM) [21,22], the immersed boundary method (IBM) [23], the boundary element method (BEM) [14], and the Voronoi interface method (VIM) [24]. In the last few decades, the numerical methods for solving interface problems have reached a certain maturity and made satisfactory progress. However, the above-mentioned methods usually require either a body-fitted or unfitted mesh to treat the interface problems, and the main difficulty lies in the body-fitted mesh generation or in the technique designed to dissect the intersecting geometry of the interface and properly discretize interface conditions. Interface problems are still challenging due to the low global regularity and irregular geometry of interfaces.

In recent years, many efforts have been made to use neural networks to solve interface problems since these methods are meshfree and can take advantage of deep learning techniques such as automatic differentiation and GPU acceleration. In particular, neural network-based approaches exhibit notable advantages in treating high-dimensional problems, inverse problems, and simultaneously solving parametric PDE problems that involve learning the solution operator (operator learning), which issues also exist in interface problems. In addition, the use of multiple neural networks based on the domain decomposition method (DDM) has attracted increasing attention as they are more accurate and flexible in dealing with the interface and have shown remarkable success in various interface problems [25–28]. This idea is further studied from the numerical aspect in our previous work [28], where the proposed interfaced neural networks are able to balance the interplay between different terms in the composite loss function and improve the performance in terms of accuracy and robustness. The above-mentioned works focus on obtaining empirical results, whereas we focus on theoretical aspects such as the convergence of PINNs for solving interface problems in this paper.

Along with the remarkable empirical achievements of deep learning methods, rigorous error and convergence analysis are also being developed and enriched. In previous work [29], the Hölder continuity constant was used to obtain the generalization analysis of PINNs in the case of linear second-order elliptic and parabolic type PDEs. [30,31] used quadrature points in the formulation of the loss and carried out an a-posteriori-type generalization error analysis of PINNs for both forward and inverse problems. [32] studied linear PDEs and proved both a priori and posterior estimates for PINNs and variational