Commun. Math. Res.
doi: 10.4208/cmr.2022-0040

Vol. **39**, No. 2, pp. 296-330
May 2023

A Structure-Preserving Numerical Method for the Fourth-Order Geometric Evolution Equations for Planar Curves

Eiji Miyazaki*, Tomoya Kemmochi, Tomohiro Sogabe and Shao-Liang Zhang

Graduate School of Engineering, Nagoya University, Furo-cho, Chikusa-ku, Nagoya, Aichi 464-8601, Japan.

Received 16 July 2022; Accepted 18 November 2022

Abstract. For fourth-order geometric evolution equations for planar curves with the dissipation of the bending energy, including the Willmore and the Helfrich flows, we consider a numerical approach. In this study, we construct a structure-preserving method based on a discrete variational derivative method. Furthermore, to prevent the vertex concentration that may lead to numerical instability, we discretely introduce Deckelnick's tangential velocity. Here, a modification term is introduced in the process of adding tangential velocity. This modified term enables the method to reproduce the equations' properties while preventing vertex concentration. Numerical experiments demonstrate that the proposed approach captures the equations' properties with high accuracy and avoids the concentration of vertices.

AMS subject classifications: 37M15, 65M06

Key words: Geometric evolution equation, Willmore flow, Helfrich flow, numerical calculation, structure-preserving, discrete variational derivative method, tangential velocity.

1 Introduction

In this paper, we design a numerical method for the Willmore and the Helfrich flows for planar curves. The Willmore flow is a geometric evolution equation that models the behavior of elastic bodies [23]. It is a gradient flow with regard to the

*Corresponding author. *Email address:* `e-miyazaki@na.nuap.nagoya-u.ac.jp` (E. Miyazaki)

bending energy

$$B(t)=\frac{1}{2}\int_{C(t)}(k-c_0)^2 ds, \tag{1.1}$$

where $C(t)$ is a closed curve, s is the arc-length parameter of $C(t)$, k is the curvature of $C(t)$, and c_0 is a given constant [6,7]. The Willmore flow is given by

$$\frac{\partial \boldsymbol{X}}{\partial t}=-\delta B, \quad \delta B=-\left\{\frac{\partial^2 k}{\partial s^2}+\frac{1}{2}(k-c_0)^3\right\}\boldsymbol{N}, \tag{1.2}$$

where δB is the gradient of B, $\boldsymbol{X}$ is a point on the closed curve $C(t)$, and $\boldsymbol{N}$ is the unit outward normal vector of $C(t)$.

The Helfrich flow is a gradient flow for the bending energy under the constraint that the length and the enclosed area of the curve are conserved. The Helfrich flow is given by

$$\begin{aligned}&\frac{\partial \boldsymbol{X}}{\partial t}=\left\{\frac{\partial^2 k}{\partial s^2}+\frac{1}{2}(k-c_0)^3+k\lambda+\mu\right\}\boldsymbol{N},\\ &\begin{pmatrix}\lambda\\ \mu\end{pmatrix}=\frac{1}{\langle k\rangle^2-\langle k^2\rangle}\begin{pmatrix}1 & -\langle k\rangle\\ -\langle k\rangle & -\langle k^2\rangle\end{pmatrix}\begin{pmatrix}\langle k\delta B\rangle+c_0^2\langle k^2\rangle/2\\ \langle \delta B\rangle+c_0^2\langle k\rangle/2\end{pmatrix},\end{aligned} \tag{1.3}$$

where

$$\langle F\rangle=\frac{1}{L}\int_{C(t)}F ds, \quad L=\int_{C(t)}ds. \tag{1.4}$$

Helfrich proposed an optimization problem that mathematically models the shape of red blood cells [4,12]. The Helfrich flow was proposed to solve this optimization problem [15,16]. Note that the Willmore and the Helfrich flows are fourth-order nonlinear evolution equations.

Some numerical approaches have been investigated for (1.2) and (1.3). There are two primary problems in the numerical computation of these flows:

- Numerical computation using general-purpose approaches (e.g. the Runge-Kutta method) can become unstable if the time step size is too large.
- When one approximates curves by polygonal curves, the vertices may be concentrated as the time step proceeds. The concentration of vertices may make numerical computations unstable.

There are some numerical methods regarding the above problems for (1.2) and (1.3). In [1,2], linear numerical schemes are proposed. It is based on the finite element method and approximates a closed curve by a closed polygonal curve. Additionally, these methods implicitly involve a tangential velocity through a mass lumped inner product. In [3], a semi-implicit numerical method for (1.2) is pro-