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A Fast Temporal Second Order Difference Scheme for the Fractional Sub-diffusion Equations on One Dimensional Space Unbounded Domain

Ren-Jun Qi and Zhi-Zhong Sun*

School of Mathematics, Southeast University, Nanjing 210096, China.

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Abstract. The numerical solution of the fractional sub-diffusion equations on one dimensional space unbounded domain is considered. Based on the high-order local artificial boundary conditions proposed in [Zhang W., *et al.*, J. Math. Study., 2017, 50(1): 28-53], the original space unbounded problem can be reformulated to an initial-boundary value problem on a bounded computational domain. By Alikhanov's $L2\text{-}1_\sigma$ formula and sum-of-exponentials approximation, a fast temporal second order difference scheme for the reduced problem is presented. The unique solvability, stability and convergence order $O(\tau^2+h^2)$ of the proposed method are proved by means of energy method, where τ and h denote the time and space step sizes, respectively. Some numerical examples are included to validate the theoretical results. To the best of our knowledge, this is the first work that combines the high order numerical method with the artificial boundary method for the time fractional diffusion problems on spatial unbounded domains.

AMS subject classifications: 65M06, 65M12, 65M15, 65M99

Key words: Fractional sub-diffusion equations, space unbounded domain, high-order local artificial boundary conditions, difference scheme, fast algorithm, energy method

1 Introduction

The fractional partial differential equations (FPDEs) have been received much attention for their effective descriptions of the anomalous diffusion phenomenon observed in some materials and processes with memory and hereditary properties [1–4]. Replacing the integer order time derivatives in classic diffusion equations by fractional derivatives in Caputo sense of order $\alpha\,(0<\alpha<2)$, the fractional sub-diffusion equations $(0<\alpha<1)$ and the fractional wave equations $(1<\alpha<2)$ are obtained, respectively [5,6].

*Corresponding author. *Email addresses:* rjqi97@163.com (Qi R J), zzsun@seu.edu.cn (Sun Z Z)

The analytic solutions of most of FPDEs are not available or they are usually associated with some specific functions including the Mittag-Leffler function, the Wright function or the hyperbolic geometry functions which bring a huge obstacle to the practical calculation. Therefore, the study of the numerical approaches becomes the main focus. For the time fractional diffusion equations, there are numerous research works about their numerical treatments and we just report some among them here. Sun and Wu [7] proved strictly the convergence order of $2-\alpha$ of the $L1$ formula and some difference schemes for the fractional sub-diffusion and wave equations were derived. Lin and Xu [8] gave an approximation for the fractional sub-diffusion equations using $L1$ formula in time and Legendre spectral methods in space. Li *et al.* [9] studied the $L1$ Galerkin finite element methods for the time fractional nonlinear parabolic equations. In [10], Alikhanov presented the $L2\text{-}1_\sigma$ formula for Caputo fractional derivative with $3-\alpha$ order and some temporal second order difference schemes were constructed for the fractional sub-diffusion equations. Combining the order reduction method, Sun *et al.* [11] developed the temporal second order difference scheme for the fractional wave equations. Due to the historical dependence, the numerical schemes mentioned above all require $O(N^2)$ computational cost with N denoting the time levels. Jiang *et al.* [12] proposed the sum-of-exponentials (SOE) approximation for the kernel $t^{-\alpha}$ in Caputo fractional derivative, by which the fast $L1$ formula was obtained which reduces the computational complexity significantly while keeping the accuracy. Yan *et al.* [13] further derived the fast $L2\text{-}1_\sigma$ formula. For more corresponding works, we can see [14, 15].

Many problems in science and engineering can be described by partial differential equations on spatial unbounded domains [16, 17], the heat transfer in solids, the flow around an airfoil and option pricing are typical examples. Due to the unboundedness of the physical area, some traditional numerical methods can not be applied directly. In the past four decades, the artificial boundary methods (ABMs) have become efficient numerical methods for this kind of problems. They introduce the artificial boundaries to divide the domain into the bounded computational part and the remaining unbounded part, and impose some suitable artificial boundary conditions (ABCs) on the artificial boundaries. Then the original problem on space unbounded domains is transformed into the boundary or initial-boundary value problem (called "reduced problem") on space bounded domain, which can be solved by many common numerical algorithms such as the finite difference method or the finite element method. We shall refer the readers to [18] for the comprehensive description of ABMs.

Roughly speaking, the ABCs can be divided into the global ABCs and the local ABCs. For most space unbounded problems, the exact ABCs are the global ABCs and usually take the form of integrals containing the unknown function with its derivatives. Until now, many scholars have studied the exact global ABCs for various spatial unbounded problems and constructed some valid numerical methods for the reduced problems, such as Poisson equations [19], heat equations [16, 20], Burgers equation [21], Schrödinger equations [22], Navier-Stokes system [23,24], etc. The reduced problems with exact global ABCs are well-posed but require large computational cost and memory since the function