J. Math. Study
doi: 10.4208/jms.v56n3.23.04

Vol. **56**, No. 3, pp. 309-326
September 2023

System of Set-Valued Nonlinear Generalized Variational Inclusion Problems in Banach Spaces

Sumeera Shafi*

Department of Mathematics, S.P. College, Cluster University, Srinagar 190001, India.

Received May 20, 2022; Accepted February 1, 2023;
Published online July 20, 2023.

Abstract. In this paper, we consider a system of set-valued nonlinear generalized variational inclusion problems (SNGVIP) in q-uniformly smooth Banach spaces. We define a class of $H(\cdot,\cdot)$-mixed mappings and its associated class of resolvent operators. Further, using proximal-point mapping method, we suggest an iterative algorithm for finding the solution of the system of set-valued nonlinear generalized variational inclusion problems. Furthermore, we discuss the convergence criteria of the sequences generated by the iterative algorithm.

AMS subject classifications: 47H09, 47J20, 49J40.

Key words: System of set-valued nonlinear generalized variational inclusion problems, $H(\cdot,\cdot)$-mixed mappings, proximal-point mapping, q-uniformly smooth Banach spaces, convergence analysis.

1 Introduction

Variational inclusion problems are among the most interesting and intensively studied classes of mathematical problems and have wide applications in the fields of optimization and control, economics and transportation equilibrium, engineering science. For the past years, many existence results and iterative algorithms for various variational inequality and variational inclusion problems have been studied. For details, please refer [1-8,14,15,17,18,20,21] and the references therein. Zou and Huang [22,23] introduced and studied $H(\cdot,\cdot)$-accretive mappings, Kazmi *et al.* [9-11] introduced and studied generalized $H(\cdot,\cdot)$-accretive mappings, $H(\cdot,\cdot)-\eta$-proximal-point mappings. In 2011, Li and Huang [12] studied the graph convergence for the $H(\cdot,\cdot)$-accretive mapping and showed the equivalence between graph convergence and proximal-point mapping convergence for the $H(\cdot,\cdot)$-accretive mapping sequence in a Banach space.

*Corresponding author. *Email address:* sumeera.shafi@gmail.com (Shafi S)

Motivated and inspired by the above works and by the ongoing research in this direction, in this paper, we introduce and study a system of set-valued nonlinear generalized variational inclusion problems involving $H(\cdot,\cdot)$-mixed mappings, a natural generalization of accretive(monotone) mappings in q-uniformly smooth Banach spaces. Using proximal-point mapping method, we suggest an iterative algorithm for solving the system. Furthermore, we prove that the sequences generated by the algorithm converge strongly to a solution of the system.

2 Proximal-point mapping and formulation of problem

We need the following definitions and results from the literature.

Let X be a real Banach space equipped with norm $\|\cdot\|$ and $X^\star$ be the topological dual space of X. Let $<\cdot,\cdot>$ be the duality pairing between X and $X^\star$ and 2^X be the power set of X.

Definition 2.1 ([19]). *For $q>1$, a mapping $J_q:X\to 2^{X^\star}$ is said to be generalized duality mapping, if it is defined by*

$$J_q(x)=\{f\in X^\star:\langle x,f\rangle=\|x\|^q,\|x\|^{q-1}=\|f\|\},\qquad \forall x\in X.$$

In particular, J_2 is the usual normalized duality mapping on X, given as

$$J_q(x)=\|x\|^{q-2}J_2(x),\qquad \forall x(\neq 0)\in X.$$

Note that if $X\equiv H$, a real Hilbert space, then J_2 becomes the identity mapping on X.

Definition 2.2 ([19]). *A Banach space X is said to be smooth if, for every $x\in X$ with $\|x\|=1$, there exists a unique $f\in X^\star$ such that $\|f\|=f(x)=1$.*

The modulus of smoothness of X is the function $\rho_X:[0,\infty)\to[0,\infty)$, defined by

$$\rho_X(\sigma)=\sup\left\{\ \frac{\|x+y\|+\|x-y\|}{2}-1:x,y\in X,\|x\|=1,\|y\|=\sigma\ \right\}$$

Definition 2.3 ([19]). *A Banach space X is said to be*

(i) uniformly smooth if $\lim_{\sigma\to 0}\dfrac{\rho_X(\sigma)}{\sigma}=0$,

(ii) q-uniformly smooth, for $q>1$, if there exists a constant $c>0$ such that $\rho_X(\sigma)\le c\sigma^q$, $\sigma\in[0,\infty)$.

Note that if X is uniformly smooth, J_q becomes single-valued.

Lemma 2.1 ([19]). *Let $q>1$ be a real number and let X be a smooth Banach space. Then the following statements are equivalent:*

(i) X is q-uniformly smooth.