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A Two-Dimensional Third-Order CESE Scheme for Ideal MHD Equations

Yufen Zhou[1,*] and Xueshang Feng[1,2]

[1] *SIGMA Weather Group, State Key Laboratory for Space Weather, National Space Science Center, Chinese Academy of Science, Beijing 100190, China.*
[2] *Shenzhen Key Laboratory of Numerical Prediction for Space Storm, Institute of Space Science and Applied Technology, Harbin Institute of Technology, Shenzhen 518055, China.*

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Abstract. In this paper, we construct a two-dimensional third-order space-time conservation element and solution element (CESE) method and apply it to the magnetohydrodynamics (MHD) equations. This third-order CESE method preserves all the favorable attributes of the original second-order CESE method, such as: (i) flux conservation in space and time without using an approximated Riemann solver, (ii) genuine multi-dimensional algorithm without dimensional splitting, (iii) the use of the most compact mesh stencil, involving only the immediate neighboring cells surrounding the cell where the solution at a new time step is sought, and (iv) an explicit, unified space-time integration procedure without using a quadrature integration procedure. In order to verify the accuracy and efficiency of the scheme, several 2D MHD test problems are presented. The result of MHD smooth wave problem shows third-order convergence of the scheme. The results of the other MHD test problems show that the method can enhance the solution quality by comparing with the original second-order CESE scheme.

AMS subject classifications: 65M08, 76W05

Key words: CESE method, third-order, MHD equations.

1 Introduction

The space-time conservation element and solution element (CESE) method was originally proposed by Chang and co-workers [6,7] for solving conservation laws. In contrast to conventional finite volume method (FVM) and finite difference method (FDM), the CESE method has several unique features. It treats space and time in a unified manner

*Corresponding author. *Email addresses:* `yfzhou@swl.ac.cn` (Y. Zhou), `fengx@swl.ac.cn` (X. Feng)

when imposing local and global space-time flux conservation. There is no need to employ the reconstruction or Riemann solver. The space-time domain is divided into space-time Solution Elements (SEs), in which the primary unknowns and the fluxes are discretized and represented by simple smooth functions. The space-time domain is also divided into non-overlapping space-time Conservation Elements (CEs), over which flux conservation is enforced in both space and time. It has the most compact stencil. Only the immediate neighboring mesh cells of the solution point are involved in the computational algorithm. It achieves the same accuracy in time and space with a fully discrete one-stage formulation. Owing to its numerical accuracy and robustness, the CESE method has been successfully extended and applied to compute Euler, e.g., [3,4,36], Navier-Stokes, e.g., [10,15,27], and magnetohydrodynamic (MHD) equations, e.g., [13,14,22,33].

However, the original CESE scheme [6,7] cannot be directly applied in the viscous flow and inviscid flow problems with shocks due to its non-dissipative property. To overcome the shortcoming, Zhang *et al.* [36] proposed its dissipative extension for solving the unsteady Euler equations. But it is sensitive to the local Courant Friedrichs Lewy (CFL) number. To overcome this limitation, a Courant number insensitive (CNI) CESE scheme is proposed to adjust the dissipation via the local CFL number [8,30,34,35]. Later, by introducing approximate Riemann solvers or other upwind techniques to compute the flux vector at the interfaces between sub-CEs, Shen *et al.* [24,26] and Shen and Wen [25] proposed upwind CESE schemes for capturing contact discontinuities. Efforts have also been made to design higher-order CESE schemes. Liu and Wang [20] developed an arbitrary-order one-dimensional CESE scheme based on arbitrary Taylor expansions in the solution elements. Chang [9] proposed a highly-stable high-order CESE method for solving the one-dimensional Burgers equation. Then Bilyeu *et al.* [3,4] extended Chang's work to solve a system of linear and non-linear hyperbolic partial differential equations in one- and two-dimensions. Shen *et al.* [23] extended it to high-order versions including third and fourth order for the Euler equation on hybrid grids in two-dimensions. Yang *et al.* [33] extended the CESE MHD solver to a fourth-order version. However, the fourth-order CESE MHD solver can only be applied to the rectangular grids in Cartesian coordinate. All the boundaries of the CEs are parallel to the coordinate surfaces, and the normal direction is along the coordinate axis.

Moreover, so far, there is no detailed derivation of the third-order accuracy CESE method for MHD equations. In the present study, we extend the second-order CESE method to third orders for 2D MHD equations and report detailed derivation. Moveover, the third-order CESE scheme can be directly applied to the unstructured meshes. The third-order CESE method preserves all the features of the original second-order CESE method. It can provide more accurate solutions. For testing the accuracy, resolution, and efficiency of the third-order CESE method, we simulate several 2D MHD benchmark problems, such as smooth Alfv*é*n wave problem, oblique shock tube problem, Orszag-Tang vortex and rotor problem.

The paper is organized as follows. Section 2 illustrates the 2D MHD governing equations. Section 3 presents the CESE method for calculating the flow variables. Section 4