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A Quadratic Serendipity Finite Volume Element Method on Arbitrary Convex Polygonal Meshes

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Abstract. Based on the idea of serendipity element, we construct and analyze the first quadratic serendipity finite volume element method for arbitrary convex polygonal meshes in this article. The explicit construction of quadratic serendipity element shape function is introduced from the linear generalized barycentric coordinates, and the quadratic serendipity element function space based on Wachspress coordinate is selected as the trial function space. Moreover, we construct a family of unified dual partitions for arbitrary convex polygonal meshes, which is crucial to finite volume element scheme, and propose a quadratic serendipity polygonal finite volume element method with fewer degrees of freedom. Finally, under certain geometric assumption conditions, the optimal H^1 error estimate for the quadratic serendipity polygonal finite volume element scheme is obtained, and verified by numerical experiments.

AMS subject classifications: 65N08, 65N12

Key words: Quadratic serendipity polygonal finite volume element method, arbitrary convex polygonal meshes, Wachspress coordinate, unified dual partitions, optimal H^1 error estimate.

1 Introduction

Due to the local conservation property, finite volume element method (FVEM) is a popular method for numerical simulation, and has been researched extensively in the last decades (e.g. [1–18]). At present, the linear FVEM on triangular and quadrilateral meshes has a relatively complete theory system, such as coercivity results, H^1 and L^2 error estimates. However, the research of quadratic FVEM is more difficult than linear scheme, then we give a brief review for the development of quadratic FVEM. For the quadratic FVEM on triangular meshes, there are plenty of schemes (e.g. [1–6])

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based on different dual partitions with two parameters α and β. The coercivity of these schemes are almost established under certain minimum angle conditions, but up to now, only one scheme proposed in [5] is unconditionally stable. Moreover, for the existing schemes, there is only one scheme which has been proved that it has both the optimal H^1 and L^2 error estimates in [4]. Based on the existing coercivity results for the quadratic scheme in [4], it is not unconditionally stable, and how to remove the minimum angle condition is still an open and expectant problem. For the quadratic FVEM on quadrilateral meshes, the scheme [7] obtained the optimal convergence rate in H^1 norm for rectangular meshes, and the scheme [8] reached similar result for h^2-parallelogram meshes. But L^2 convergence rates of them are all not optimal. By using the optimal stress points to construct dual partition, [9, 10] obtained the optimal convergence rates in L^2 norm on rectangular meshes, and the results were extended to h^2-parallelogram meshes in [11]. After that, according to the research for any order finite volume element schemes on quadrilateral meshes [12, 13], we can find the theoretical analysis of quadratic FVEM was further improved for $h^{1+\gamma}$-parallelogram meshes. In addition, recent research has extended the quadratic FVEM to tetrahedral and cuboid meshes for three-dimensional problems (e.g. [16, 17]).

For 2D problem, it's worth noting that all the above finite volume element schemes are constructed on triangular and quadrilateral meshes. We know more and more numerical schemes are constructed on polygonal and polyhedral meshes (e.g. [19–23]), however, there is little research on finite volume element scheme over polygonal meshes currently. Recently, a linear FVEM for arbitrary convex polygonal meshes was proposed in [18], and the optimal H^1 error estimate was obtained under some geometric assumptions and coercivity assumptions. To the best of our knowledge, no previous study presented the quadratic FVEM on polygonal meshes. Therefore, the focus of our research interest is constructing and analyzing the quadratic FVEM over arbitrary convex polygonal meshes.

When we obtain shape functions with quadratic precision by the pairwise products of generalized barycentric coordinates over a general polygonal cell, we find it will lead into some additional degrees of freedom inside the cell. However, the interior unknowns will produce great difficulties for constructing unified dual partitions on arbitrary polygonal meshes, and maybe it can only be done case by case (e.g. triangular meshes, quadrilateral meshes, pentagonal meshes …). Fortunately, we find some research about quadratic serendipity element shape functions over arbitrary polygonal meshes. The authors of [24] established the quadratic serendipity finite element method on polygonal meshes by using generalized barycentric coordinates, and what was more, other constructions of quadratic serendipity finite element method can be found in [25–28]. By using mean value coordinate, [29] offered a simple and efficient method to explicitly construct quadratic serendipity element shape functions on both convex and concave polygons. From the above research results, the quadratic serendipity elements only need $2n$ basis functions for n-sided polygon, associated with vertices and edge midpoints to achieve quadratic precision. The number of basis functions reduces from $n(n+1)/2$ to $2n$, which contributes to establish unified quadratic FVEM.