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Life-Span of Classical Solutions to a Semilinear Wave Equation with Time-Dependent Damping

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Abstract. This paper is concerned with the Cauchy problem for a semilinear wave equation with a time-dependent damping. In case that the space dimension $n=1$ and the nonlinear power is bigger than 2, the life-span $\widetilde{T}(\varepsilon)$ and global existence of the classical solution to the problem has been investigated in a unified way. More precisely, with respect to different values of an index K, which depends on the time-dependent damping and the nonlinear term, the life-span $\widetilde{T}(\varepsilon)$ can be estimated below by $\varepsilon^{-\frac{p}{1-K}}$, $e^{\varepsilon^{-p}}$ or $+\infty$, where ε is the scale of the compact support of the initial data.

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1 Introduction

Consider the following Cauchy problem for semilinear wave equations with time-dependent damping in n (≥ 1) space dimensions:

$$\begin{cases} \Box u(t,x)+b(t)u_t(t,x)=u^{1+p}(t,x), & (t,x)\in\mathbb{R}_+\times\mathbb{R}^n, \\ u(0,x)=\varepsilon f(x), & x\in\mathbb{R}^n, \\ u_t(0,x)=\varepsilon g(x), & x\in\mathbb{R}^n, \end{cases} \tag{1.1}$$

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where

$$\Box=\frac{\partial^2}{\partial t^2}-\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} \tag{1.2}$$

is the wave operator, $b(t)=\mu(1+t)^{-\alpha}$, $\alpha\in\mathbb{R}$, $\mu\geq 0$, $p\geq 1$ is an integer, $\varepsilon>0$ is a small parameter and

$$f(x),\, g(x)\in C_0^{\infty}(\mathbb{R}^n). \tag{1.3}$$

Eq. (1.1) can be used to describe the telegraph equation, the elastic vibration with damping proportional to the velocity and the heat conduction with finite speed of propagation etc, see [1]. The term $b(t)u_t$ is called the damping term, which prevents the motion of the wave and reduces its energy, and the coefficient $b(t)$ represents the strength of the damping. From the mathematical point of view, it is an interesting problem to study how the damping term affects the properties of the solution.

The Cauchy problem for the semilinear wave equations with **linear dissipation** (i.e. $\mu=1$ and $\alpha=0$ in (1.1)) has been studied intensively in the past three decades. For example, when $n(\leq 2)$, under some sign condition on the initial data, if the power p is smaller than $\frac{2}{n}$, Li and Zhou [1] gave an upper bound for the life-span of the *mild solutions* to the problem. While $n\geq 3$, in 2001, Todorova and Yordanov [2] studied the question of global existence, blowup and asymptotic behavior as $t\to\infty$ for the *weak solutions* of the problem. They determined a critical exponent $p_c(n)=\frac{2}{n}$, which means if $p_c(n)<p$, then all small data solutions of the problem are global, while if $p<p_c(n)$ all solutions of the problem with data positive on average blow up in finite time regardless of the smallness of the data. Using the testing function argument, Yordanov and Zhang [3] pointed out that the critical index $p_c(n)$ also belongs to the blowup scope. Li ([4]) obtained the estimates on the lower bound for the life-span of the classical solutions to a fully nonlinear wave equation with linear dissipation. Moreover, as an application of the main theorem in [1], it can be shown that Li's estimates in [4] are sharp. The method used in [4] is the global iteration method, which was first put forward by Li T. T., Chen Y. M., Yu X., and Zhou Y. ([5]- [15]) to study the life-span $\widetilde{T}(\varepsilon)$ of classical solutions to the following Cauchy problem for any given space dimension $n\geq 1$ and integer $\gamma\geq 1$

$$\begin{cases} u_{tt}-\Delta u=F(u,Du,D_xDu), & \text{in } (0,\infty)\times\mathbb{R}^n,\\ u(0,x)=\varepsilon f(x), & x\in\mathbb{R}^n,\\ u_t(0,x)=\varepsilon g(x), & x\in\mathbb{R}^n, \end{cases} \tag{1.4}$$

where

$$D_x=\left(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_n}\right), \qquad D=\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_n}\right). \tag{1.5}$$

Let

$$\hat{\lambda}=(\lambda;\,(\lambda_i),\, i=0,1\cdots,n;\,(\lambda_{ij}),\, i,j=0,1\cdots,n,\, i+j\geq 1). \tag{1.6}$$