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Positive Solutions of Second-order Difference Equation with Variable Coefficient on the Infinite Interval*

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Abstract In this paper, based on the one-signed Green's function and the compact results on the infinite interval, we obtain the existence and multiplicity of positive solutions for the boundary value problems

$$\begin{cases} \Delta^2 x(n-1) - p(n)\Delta x(n-1) - q(n)x(n-1) + f(n, x(n)) = 0, \ n \in \mathbb{N}, \\ \alpha x(0) - \beta \Delta x(0) = 0, \qquad \lim_{n\to\infty} x(n) = 0 \end{cases}$$

by the fixed point theorem in cones. The main results extend some results in the previous literature.

Keywords Positive solution, Green's function, Compact, Infinite interval.

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1. Introduction

The continuous boundary value problem on the half-line occur in the mathematical modeling of various applied problems, for example, discussion on electrostatic probe measurements in solid-propellant rocket exhausts [11], analysis of the mass transfer on a rotating disk in a non-Newtonian fluid, heat transfer in the radial flow between parallel circular disks [13] and investigation of temperature distribution in the problem of phase change of solids with temperature-dependence thermal conductivity [13]. Hence, the existence of positive solutions to the infinite interval boundary value problem of second-order ordinary differential equations have been studied by many authors (see [3, 5–7, 9, 12, 15, 16] and their references). However, the existence and multiplicity of the positive solutions to second-order difference equations on the half-line have only few results such as [1, 2, 8, 14].

Let $\mathbb{N} = \{1, 2, 3, \cdots\}$, $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$, $N(a,\ b) = \{a, a+1, \cdots, b\}$, for $a < b$.

In 2001, Agarwal et al., [2] studied the positive solutions of the following bound-

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ary value problem on the infinite interval

$$\begin{cases} \Delta^2 x(n-1) + f(n, x(n)) = 0, & n \in \mathbb{N} \\ x(0) = 0, & \lim_{n\to\infty} x(n) = \gamma \in \mathbb{R} \end{cases} \tag{1.1}$$

by employing upper and lower solution methods. In 2006, Tian and Ge [14] obtained the existence of multiple positive solutions for the problem

$$\begin{cases} \Delta^2 x(n-1) - p\Delta x(n-1) - qx(n-1) + f(n, x(n)) = 0, \ n \in \mathbb{N}, \\ \alpha x(0) - \beta \Delta x(0) = 0, \qquad \lim_{n\to\infty} x(n) = 0, \end{cases} \tag{1.2}$$

where $p,\ \alpha,\ \beta \ge 0,\ \alpha^2 + \beta^2 > 0,\ q > 0,\ 1 + p > q$, and $f : \mathbb{N} \times [0, \infty) \to [0, \infty)$ are continuous. The main proofs are based on the fixed point theorem in Fréchet space.

Definitely, the natural question is whether or not the positive solution of problem (1.2) on the infinite interval exists in the Banach space. The key points are the compact results on the infinite interval and the one-signed Green's function of (1.2) and its bounded properties. This is an interesting problem which is different from the properties of Green's function on finite interval.

Motivated by what has been mentioned above, we discuss the one-signed property of Green's function and its bounded properties, and obtain the existence and multiplicity of positive solutions of the following problem

$$\begin{cases} \Delta^2 x(n-1) - p(n)\Delta x(n-1) - q(n)x(n-1) + f(n, x(n)) = 0, \ n \in \mathbb{N}, \\ \alpha x(0) - \beta \Delta x(0) = 0, \qquad \lim_{n\to\infty} x(n) = 0, \end{cases} \tag{1.3}$$

where $p : \mathbb{N} \to [0,\ \infty)$, $q : \mathbb{N} \to (0,\ \infty)$ are bounded functions, $\alpha,\ \beta \ge 0,\ \alpha^2+\beta^2 > 0$ and $f : \mathbb{N} \times [0, \infty) \to [0, \infty)$ are continuous.

Notice that (1.3) generalizes (1.2). It is worth pointing out that Green's function of the associated linear problem

$$\begin{cases} \Delta^2 x(n-1) - p(n)\Delta x(n-1) - q(n)x(n-1) = 0, \ n \in \mathbb{N}, \\ \alpha x(0) - \beta \Delta x(0) = 0, \qquad \lim_{n\to\infty} x(n) = 0 \end{cases} \tag{1.4}$$

cannot be explicitly expressed by elementary functions. These make our approach more difficult. Fortunately, we find Perron's theorem [8] and the compact theorem in the Banach space $l_\diamond = \{x(\cdot) \in l^\infty(\mathbb{N}_0) \,|\, \lim_{n\to\infty} x(n) = x(\infty)\}$ [10] to overcome these difficulties.

The rest of this paper is arranged as follows. In Section 2, we construct the Green's function of (1.4), and prove its one-signed and bounded properties. In Section 3, we state the compact theorem on the infinite interval and the transfer problem (1.3) to the compact summing operator in Banach space $l_\diamond$. In Section 4, we give the existence and multiplicity results of positive solutions for problem (1.3).

Throughout this paper, we denote the summation of $x(n)$ from $n = a$ to $n = b$ by $\sum_{n=a}^{b} x(n)$ with the understanding that $\sum_{n=a}^{b} x(n) = 0$ for all $a > b$, and the product of $x(n)$ from $n = a$ to $n = b$ by $\prod_{n=a}^{b} x(n)$ with the understanding that $\prod_{n=a}^{b} x(n) = 1$ for all $a > b$.