Journal of Nonlinear Modeling and Analysis
Volume 4, Number 4, December 2022, 686–700
http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2022.686

Triple Positive Solutions of Boundary Value Problems for High-order Fractional Differential Equation at Resonance with Singularities*

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Abstract In this paper, we investigate the existence of triple positive solutions of boundary value problems for high-order fractional differential equation at resonance with singularities by using the fixed point index theory and the Leggett-Williams theorem. The spectral theory and some new height functions are also employed to establish the existence of triple positive solutions. The nonlinearity involved is arbitrary fractional derivative, and permits singularity.

Keywords Triple positive solution, Fractional differential equation, Resonance, Singularity.

MSC(2010) 34B18, 34A08, 34F15, 35A21.

1. Introduction

In this paper, we consider the following boundary value problem for high-order fractional differential equation (FBVP for short) at resonance

$$\begin{cases} D_{0+}^{\alpha}x(t)+f(t,x(t),D_{0+}^{\beta}x(t))=0,\ 0<t<1,\\ x(0)=x'(0)=x''(0)=\cdots=x^{(n-2)}(0)=0,\ D_{0+}^{\beta}x(1)=\lambda\int_0^{\eta}m(t)D_{0+}^{\beta}x(t)dt, \end{cases} \tag{1.1}$$

where D_{0+}^{α} is the standard Riemann-Liouville derivative, $m\in L[0,1]$ is nonnegative and may be singular at $t=0$ and $t=1$, $n-1<\alpha<n$, $n\geq 3$, $\alpha-(n-1)>\beta>0$, $0<\eta\leq 1$, $\lambda\int_0^{\eta}m(t)t^{\alpha-\beta-1}dt=1$, and the nonlinearity $f(t,x,y)$ permits singularities at $t=0,1$ and $x=y=0$.

We note that (1.1) happens to be at resonance in the case that the corresponding homogeneous boundary value problem

$$\begin{cases} D_{0+}^{\alpha}x(t)=0,\ 0<t<1,\\ x(0)=x'(0)=x''(0)=\cdots=x^{(n-2)}(0)=0,\ D_{0+}^{\beta}x(1)=\lambda\int_0^{\eta}m(t)D_{0+}^{\beta}x(t)dt, \end{cases} \tag{1.2}$$

has a solution $ct^{\alpha-1}, c\in\mathbb{R}, c\neq 0$, as a nontrivial solution. That is, the derivative operator in the boundary value problem is not invertible.

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*The authors were supported by the Natural Science Foundation of China (Grant No. 62073153) and Shandong Provincial Natural Science Foundation (Grant No. ZR2020MA016).

The research on the boundary value problem of fractional differential equation is mainly carried out to extend some effective methods in integer differential equation such as special function method, Laplace transform, Fourier transform, iterative method, operator calculus, combination method and fixed point theorem (see [3, 13, 18, 22]). The main results focus on the linear non-resonant boundary value problem (see [1, 21, 23–27]). The study on the resonant boundary value problem is not perfect.

The resonant boundary value problem is that the homogeneous problem corresponding to the equation has a nontrivial solution. That is to say, the derivative operator in the boundary value problem is not invertible. Boundary value problem at resonance, as a kind of special boundary value problem in differential equations, has a very wide application prospect in the fields of celestial mechanics, aerodynamics, material mechanics, fluid mechanics and so on (see [2, 6, 9, 12, 16]).

For integer-order boundary value problems at resonance, the methods used by researchers generally include topological degree theory, Mawhin's overlap degree theorem, fixed point theorem, critical point theorem, function analysis theory, phase plane analysis method and so on (see [4, 5, 7, 10, 17, 20]). Compared with the boundary value problem of integer-order differential equations at resonance, the study of fractional order started late. The first one was Kosmatov's application of Mawhin's continuity theorem to study the following fractional-order three-point boundary value problem at resonance (see [8])

$$\begin{cases} D_{0+}^{\alpha}u(t) = f(t, u(t), u'(t)) = 0, \ 0 < t < 1, \\ D_{0+}^{\alpha-2}u(0) = 0, \ \eta u(\xi) = u(1). \end{cases}$$

In most cases of the real life, it is necessary to solve the positive solution of the differential equation under the boundary value conditions, which requires sufficient theoretical proof. For the boundary value problems of integer or fractional-order differential equations at resonance, it is not difficult to find that no matter using Mawhin's overlap degree theorem or generalized continuity theorem by Professor Ge, we can only obtain the existence of solutions, but cannot guarantee the existence of positive solutions. The study on the positive solution of boundary value problems for fractional-order differential equations at resonance has only been paid attention to by researchers in the recent years. There are only a few pieces of literature (see [5, 8, 14, 15, 17, 20]). Yang and Wang applied Leggett-Williams fixed point theorem to study the result that the fractional boundary value problem at resonance has at least one positive solution (see [19])

$$\begin{cases} D_{0+}^{\alpha}u(t) = f(t, u(t)) = 0, \ 0 < t < 1, \\ u(0) = 0, \ \eta u'(0) = u'(1). \end{cases}$$

As far as we know, triple positive solutions of boundary value problems for high-order fractional differential equation at resonance with singularities has not been considered. Inspired by the work above, we aim to fill this gap. This paper is organized as follows. First, we reduce non-perturbed boundary value problems at resonance to the equivalent non-resonant perturbed problems with the same boundary conditions. Then, we derive the Green's function and corresponding properties. Finally, the existence of triple positive solutions is obtained by using the Leggett-Williams theorem and the fixed point index theory.