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Existence and Uniqueness of Solutions for the Initial Value Problem of Fractional q_k-Difference Equations for Impulsive with Varying Orders*

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Abstract The paper studies the existence and uniqueness for impulsive fractional q_k-difference equations of initial value problems involving Riemann-Liouville fractional q_k-integral and q_k-derivative by defining a new q-shifting operator. In this paper, we obtain existence and uniqueness results for impulsive fractional q_k-difference equations of initial value problems by using the Schaefer's fixed point theorem and Banach contraction mapping principle. In addition, the main result is illustrated with the aid of several examples.

Keywords Impulsive fractional q_k-difference equation, Boundary value problem, Existence, Uniqueness.

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1. Introduction

Fractional calculus is a relatively new research field, and it can describe certain phenomena as well, which has attracted increasing attention in recent years. The quantum calculus is known as the calculus without limits. It substitutes the classical derivative by a difference operator, which allows one to deal with sets of nondifferentiable functions. Quantum difference operators appear in several branches of mathematics, i.e., basic hype-geometric functions, combinatorics, the theory of relativity. For the fundamental concepts of quantum calculus, we refer to the reader to the work by Kac and Cheung [5,8].

In the recent years, the topic of q-calculus has attracted the attention of several researchers, and a variety of new results can be found in the papers [3,4,7,9,10,12–14,22] and the references therein. In real life, there are many processes and phenomena that are characterized by rapid changes in their state. We usually keep things instantaneous mutations occurred in the process of its development called impulsive phenomena. The phenomenon has been widely appearing in all fields of production and technology research. The most prominent feature of impulsive differential equations is taking the influence of the condition of sudden and abrupt phenomenon into full consideration. It has been extensively used in population

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ecological dynamics systems, infectious disease dynamics as well as descriptions of phenomenon like disease, harvesting and so on.

Impulsive differential equations, in other words, differential equations involving impulsive factors, appear as a natural description of observed evolution phenomenon of several real world problems. For some monographs on impulsive differential equations, we refer to [6, 20, 21].

In [16], the notions of q_k-integral of a function $f : J_k := [t_k, t_{k+1}] \to \mathbb{R}$ have been introduced, and their basic properties were proved and applied, Tariboon et al., investigated the first and second-order initial value problems of impulsive q_k-difference equation respectively, as shown below

$$\begin{aligned}
&D^2_{q_k} x(t) = f(t, x(t)),\ t \in J,\ t \neq t_k,\\
&\Delta x(t_k) = I_k(x(t_k)),\ k = 1, 2, \cdots m,\\
&D_{q_k} x(t_k^+) - D_{q_{k-1}} x(t_k) = I_k^*(x(t_k)),\ k = 1, 2, \cdots m,\\
&x(0) = \alpha,\ D_{q_0} x(0) = \beta,\\
&D_{q_k} x(t) = f(t, x(t)),\ t \in J,\ t \neq t_k,\\
&\Delta x(t_k) = I_k(x(t_k)),\ k = 1, 2, \cdots m,\\
&x(0) = x_0,
\end{aligned}$$

where $x_0 \in \mathbb{R}, \alpha, \beta \in \mathbb{R}, 0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = T, f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $I_k \in C(\mathbb{R}, \mathbb{R}), \Delta x(t_k) = x(t_k^+) - x(t_k), k = 1, 2, \cdots m$ and $0 < q_k < 1$ for $k = 0, 1, 2, \cdots m$. In addition, q_k-calculus analogues of some classical integral inequalities, such as Hölder, Hermite-Hadamard, Trapezoid, Ostrowski, Cauchy-Bunyakovsky-Schwarz, Grüss and Grüss-Cebysev, were proved in [17].

In 2015, Agarwal et al., [2] investigated the existence of positive solutions for nonlinear impulsive q_k-difference equations via a monotone iterative method

$$\begin{aligned}
&D_{q_k} u(t) = f(t, u(t)),\ 0 < q_k < 1,\ t \in J',\\
&\Delta u(t_k) = I_k(u(t_k)),\ k = 1, 2, \cdots m,\\
&u(0) = \lambda u(\eta) + d,\ \eta \in J_r,\ r \in \mathbb{Z},
\end{aligned} \tag{1.1}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R}^+),\ I_k \in C(\mathbb{R}, \mathbb{R}^+),\ J = [0, T],\ T > 0,\ 0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = T,\ J' = J \setminus \{t_1, t_2, \cdots t_m\},\ J_r = (t_r, T),\ 0 \leq \lambda < 1,\ 0 \leq r \leq m$ and $\Delta u(t_k) = u(t_k^+) - u(t_k^-),\ u(t_k^+)$ and $u(t_k^-)$ denote the right and left limits of $u(t)$ at $t = t_k (k = 1, 2, \cdots m)$ respectively.

In [18], the new concepts of fractional quantum calculus were defined by introducing a new q-shifting operator. After giving the basic properties of the new q-shifting operator, the q-derivative and the q-integral were defined. New definitions of the Riemann-Liouville fractional q-integral and q-difference of an interval $[a, b]$ were given, and their properties were discussed. As applications, the authors obtained the existence and uniqueness results of initial value problems for impulsive fractional q_k-difference equations of the orders $0 < \alpha < 1$ and $1 < \alpha < 2$ respectively, as shown below,

$$\begin{aligned}
&{}_{t_k}D^\alpha_{q_k} x(t) = f(t, x(t)),\ t \in J,\ t \neq t_k,\\
&\widetilde{\Delta}\, x(t_k) = \varphi_k(x(t_k)),\ k = 1, 2, \cdots m,\\
&x(0) = 0,\\
&{}_{t_k}D^\alpha_{q_k} x(t) = f(t, x(t)),\ t \in J,\ t \neq t_k,
\end{aligned}$$