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Existence of Nonoscillatory Solutions for a Rational Difference Equation of Higher Order*

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Abstract In this paper, we investigate a rational difference equation of higher order for the existence of nonoscillatory solutions. To prove our main results, we use an inclusion theorem stated and proved in [4]. In this way, we give an answer of an open problem formulated in [3].

Keywords Rational difference equation, Nonoscillatory solution, Appropriate equation, Inclusion theorem.

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1 Introduction and preliminaries

In their paper, Amleh et al., [2] investigated the global stability, boundedness and periodicity of the positive solutions for the following rational difference equation

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}, n = 0, 1, \cdots, \tag{1.1}$$

where α is a positive real constant, and the initial conditions x_{-1}, x_0 are positive real numbers. Under the condition that α is a negative real number and the initial conditions x_{-1}, x_0 are negative numbers, Hamza has recently studied the global stability, permanence and oscillation of equation (1.1) in [13]. However, none of the researchers above considered the existence of non-oscillatory solutions of equation (1.1).

In [11], DeVault, Kent and Kosmala considered the behavior of positive solutions to rational difference equation

$$x_{n+1} = \alpha + \frac{x_{n-k}}{x_n}, n = 0, 1, \cdots, \tag{1.2}$$

where $\alpha > 0$, and $k \in N$ is a fixed positive integer. Among other things, they have proved that all nonoscillatory solutions of equation (1.2) converge to the positive equilibrium $\bar{x} = \alpha + 1$. However, it was not shown that such solutions do exist,

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which is an interesting problem. Therefore, they presented the open problem as follows.

Open problem 1. Do there exist nonoscillatory solutions of equation (1.2)?

To the best of our knowledge, there have been no results for the above open problem. Our main aim in this note is to solve the above problem in a more general framework. More generally and precisely speaking, we will investigate the existence of nonoscillatory solutions for the following higher-order rational difference equation

$$x_{n+1} = \alpha + \frac{x_{n-k}}{\sum_{i=0}^{k-1} b_i x_{n-i}}, n = 0, 1, \cdots, \tag{1.3}$$

where α is a positive real number, $k \in N$ is a fixed positive integer, $b_0 > 0, b_i \geq 0, i = 1, 2, \cdots k-1$, and the initial values $x_{-k}, x_{-k+1}, \cdots, x_{-1}, x_0$ are nonnegative real numbers such that $\sum_{i=0}^{k-1} b_i x_{-i} > 0$. Without loss of generality, we may assume $\sum_{i=0}^{k-1} b_i = 1$.

It is easy to see that equation (1.3) with the equilibrium point $\bar{x} = \alpha + 1$ contains equation (1.1) and equation (1.2) as its special cases.

Rational difference equation is a kind of typical nonlinear difference equation, which has always been a hot spot among the subjects studied in recent years. It is more important for one to find some prototypes for the development of the basic theory of the global behavior of nonlinear difference equations of order greater than one comes from the results for rational difference equations. For the systematical investigations of this aspect, the monographs [1, 17, 18], the papers [3, 6, 8, 12–16, 20–22, 24, 25, 30] and the references cited therein are hereby referred to.

Berg's inclusion theorem [4] is the main tool to prove our main results in this paper. For its proof, we refer the reader to [5].

The paper is organized as follows. In Section 2, we give some auxiliary results needed for the proof of our main results. In Section 3, we formulate and prove our main results. In Section 4, we give examples for our main result. In Section 5, concluding remarks are given.

2 Auxiliary results

Consider the following general real nonlinear difference equation with order $m \geq 1$,

$$F(x_n, x_{n+1}, \cdots, x_{n+m}) = 0, \tag{2.1}$$

where $F : R^{m+1} \mapsto R$, $n \in N_0$.

Suppose that φ_n and ψ_n are two consequences satisfying $\psi_n > 0$ and $\psi_n = o(\varphi_n)$ as $n \to \infty$. Then, (maybe under certain additional conditions), for any given $\epsilon > 0$, there are a solution $\{x_n\}_{n=-1}^{\infty}$ of equation (2.1) and an $n_0(\epsilon) \in N$ such

$$\varphi_n - \epsilon\psi_n \leq x_n \leq \varphi_n + \epsilon\psi_n, \quad n \geq n_0(\epsilon). \tag{2.2}$$

Define

$$X(\epsilon) = \{x_n : \varphi_n - \epsilon\psi_n \leq x_n \leq \varphi_n + \epsilon\psi_n, \quad n \geq n_0(\epsilon)\},$$

which is called an asymptotic stripe. Therefore, if x_n belongs to $X(\epsilon)$, then there exists a real sequence C_n such that $x_n = \varphi_n + C_n\psi_n$ and $|C_n| \leq \epsilon$ for $n \geq n_0(\epsilon)$.

The main result in [4] is the following theorem, which is called inclusion theorem.