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Regularity Criteria to the Axially Symmetric Tropical Climate Model without Swirl

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Abstract In this paper, we consider the Cauchy problem of the axially symmetric tropical climate model with fractional dissipation. By using the energy method, we establish a new regularity criteria for the axisymmetric solutions of the 3D Tropical climate model without swirl.

Keywords Axisymmetric solution, Regularity criteria, Tropical climate model.

MSC(2010) 35B65, 35Q35, 76W05.

1. Introduction

In this paper, we consider the following 3D tropical climate model:

$$\partial_t u + (u \cdot \nabla)u - \mu\Lambda^{2\alpha}u + \nabla p + \text{div } (v \otimes v) = 0, \tag{1.1}$$

$$\partial_t v + (u \cdot \nabla)v - \nu\Lambda^{2\beta}v + \nabla\psi + (v \cdot \nabla)u = 0, \tag{1.2}$$

$$\partial_t \psi + (u \cdot \nabla)\psi - \eta\Lambda^{2\gamma}\psi + \text{div } v = 0, \tag{1.3}$$

$$\text{div } u = 0, \tag{1.4}$$

$$(u, v, \psi)(\mathbf{x}, 0) = (u_0, v_0, \psi_0), \tag{1.5}$$

where the vector fields $u(\mathbf{x}, t) = (u_1(\mathbf{x}, t), u_2(\mathbf{x}, t), u_3(\mathbf{x}, t))$ and $v(\mathbf{x}, t) = (v_1(\mathbf{x}, t), v_2(\mathbf{x}, t), v_3(\mathbf{x}, t))$ denote the barotropic mode and the first baroclinic mode of the velocity, respectively. The scalar functions $p(\mathbf{x}, t)$ and $\psi(\mathbf{x}, t)$ represent the pressure and the temperature, respectively. The fractional Laplacian operator $\Lambda = (-\Delta)^{\frac{1}{2}}$ is defined by means of the Fourier transform

$$\widehat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha \widehat{f}(\xi),$$

where $\hat{f}$ denotes the Fourier transform of f. In this paper, we set the constants $\mu = \nu = \eta = 1$.

In this paper, we study the axially symmetric solution of systems (1.1)-(1.5) without swirl ($u_\theta = 0$). Then, u, v and ψ can be rewritten as

$$u(\mathbf{x}, t) = u_r(r, z, t)e_r + u_z(r, z, t)e_z, \tag{1.6}$$

$$v(\mathbf{x}, t) = v_r(r, z, t)e_r + v_z(r, z, t)e_z, \tag{1.7}$$

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$$\psi(\mathbf{x}, t) = \psi(r, z, t). \tag{1.8}$$

Here,

$$\mathbf{x} = (x, y, z), \tag{1.9}$$

$$e_r = (\frac{x}{r}, \ \frac{y}{r}, 0), \ e_\theta = (-\frac{y}{r}, \frac{x}{r}, 0), \ e_z = (0, 0, 1), \tag{1.10}$$

$$r = \sqrt{x^2 + y^2}, \ (x, y, z) = (r\cos\theta, r\sin\theta, z). \tag{1.11}$$

By direct calculation, we obtain

$$\omega(\mathbf{x}, t) = \nabla \times u = \omega_r e_r + \omega_\theta e_\theta + \omega_z e_z,$$

where

$$\omega_r = -\partial_z u_\theta, \ \omega_\theta = \partial_z u_r - \partial_r u_z, \ \omega_z = \frac{1}{r}\partial_r(r u_\theta).$$

Now, we review some related results about the tropical climate models (1.1)-(1.5). By presenting a new quantity and utilizing a logarithmic Gronwall inequality, Li and Titi [11] established the global existence of strong solutions for systems (1.1)-(1.5) without diffusion, when $\alpha = \beta = 1$ and $\mu > 0, \ \nu > 0, \ \eta = 0$. The global well-posedness of classical solutions for the tropical climate model was obtained by Wan [16] in terms of the dissipation of the first baroclinic model of the velocity and some damping terms at small initial data. By applying the "weakly nonlinear" energy estimates, the global regularity of a tropical climate model with greatly weak dissipation of the barotropic mode was proved by Ye in [20] ($\alpha > 0, \ \beta = \gamma = 1$ and $\mu, \ \nu, \ \eta > 0$). Recently, the global regularity for the 3D tropical climate model with fractional diffusion on barotropic mode has been established by Zhu [23], when $\alpha \geq \frac{5}{2}$ and $\mu > 0, \ \nu = \eta = 0$. Then, by using the spectral analysis, the global well-posedness of the 2D viscous tropical climate model with only one damping term was proved by Ma and Wan in [14], when $\mu = \nu = 1, \ \eta = 0$. The d-dimensional system (1.1) was studied by Ma [13], and he got the local smooth solution. More studies on tropical climate models are available in [4, 5, 19, 21].

When $\psi = 0$, the tropical climate models (1.1)-(1.5) become the axisymmetric MHD system. For the axisymmetric MHD system, the global well-posedness of classical solutions was established by Lei [7]. Then, the solutions of 3D axially symmetric incompressible MHD equation was studied by Wang and Wu in [17]. Also, they established a group of global smooth solutions by using the one-dimensional solutions. Lately, the regularity criteria for the axisymmetric solutions to MHD equation was established by Li and Yuan [10], as long as $\omega_\theta \in L^q(0, T; L^p(\mathbb{R}^3))$ and $n_\theta \in L^q(0, T; L^p(\mathbb{R}^3))$ satisfy

$$\int_0^T (\|\omega_\theta\|_{L^p}^q + \|n_\theta\|_{L^p}^q)dt < \infty, \ with \ \frac{3}{p} + \frac{2}{q} \leq 2, \ \frac{3}{2} < p \leq \infty, \ 0 < q < \infty.$$

For more studies about MHD system, we can refer to [12, 15].

Systems (1.1)-(1.5) reduce to the Navier-Stokes Equations, when $\psi = v = 0$. For more studies about the axisymmetric Navier-Stokes equation, we can refer to [2, 6–9, 17, 18]. Here, we only introduce some related results. First of all, some regularity criteria about the axisymmetric weak solutions of 3D Navier-Stokes equations were established by Chae-Lee in [2]. Then, Wei [18] obtained the global regularity for the solutions of the axially symmetric Navier-Stokes system, as long as $\|r u_\theta(r, z, t)\|_{L^\infty}$