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Ground States for Singularly Perturbed Planar Choquard Equation with Critical Exponential Growth*

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Abstract In this paper, we are dedicated to studying the following singularly Choquard equation

$$-\varepsilon^2\Delta u + V(x)u = \varepsilon^{-\alpha}\left[I_\alpha * F(u)\right] f(u), \qquad x \in \mathbb{R}^2,$$

where $V(x)$ is a continuous real function on $\mathbb{R}^2$, $I_\alpha : \mathbb{R}^2 \to \mathbb{R}$ is the Riesz potential, and F is the primitive function of nonlinearity f which has critical exponential growth. Using the Trudinger-Moser inequality and some delicate estimates, we show that the above problem admits at least one semiclassical ground state solution, for $\varepsilon > 0$ small provided that $V(x)$ is periodic in x or asymptotically linear as $|x| \to \infty$. In particular, a precise and fine lower bound of $\frac{f(t)}{e^{\beta_0 t^2}}$ near infinity is introduced in this paper.

Keywords Choquard equation, critical exponential growth, Trudinger-Moser inequality, ground state solution

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1. Introduction

This paper is devoted to studying the following Choquard equation

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u = \varepsilon^{-\alpha}\left[I_\alpha * F(u)\right] f(u), & x \in \mathbb{R}^2, \\ u \in H^1(\mathbb{R}^2), \end{cases} \tag{1.1}$$

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where $\varepsilon > 0$ is a parameter, $\alpha \in (0,2)$ and $I_\alpha : \mathbb{R}^2 \to \mathbb{R}$ is the Riesz potential defined by

$$I_\alpha(x) = \frac{\Gamma\left(\frac{2-\alpha}{2}\right)}{\pi\Gamma\left(\frac{\alpha}{2}\right)2^\alpha|x|^{2-\alpha}} := \frac{A_\alpha}{|x|^{2-\alpha}}, \quad \forall\, x \in \mathbb{R}^2\backslash\{0\},$$

$F(t) = \int_0^t f(s)\mathrm{d}s$, $V \in \mathcal{C}(\mathbb{R}^2,(0,\infty))$ and $f : \mathbb{R} \to \mathbb{R}$ satisfy the following basic assumptions:

(V0) $0 < \inf_{x\in\mathbb{R}^2} V(x) := V_0 \le V(x) \le \sup_{x\in\mathbb{R}^2} V(x) := V_\infty < \infty$;

(V1) $V(x)$ is 1-periodic in x_1, x_2;

(V2) $\inf_{x\in\mathbb{R}^2} V(x) := V_0 < V_\infty := \lim_{|x|\to\infty} V(x)$;

(F1) $f \in \mathcal{C}(\mathbb{R},\mathbb{R})$ and there exists $\beta_0 > 0$ such that

$$\lim_{|t|\to\infty} \frac{|f(t)|}{e^{\beta t^2}} = 0, \quad \text{for all } \beta > \beta_0$$

and

$$\lim_{|t|\to\infty} \frac{|f(t)|}{e^{\beta t^2}} = +\infty, \quad \text{for all } \beta < \beta_0;$$

(F2) $|f(t)| = o(|t|^{\alpha/2})$ as $|t| \to 0$.

The majority of the literature focuses on the study of equation (1.1) in $\mathbb{R}^N (N \ge 3)$. Let us recall some of them as follows. The singularly perturbed elliptic equation

$$-\varepsilon^2\Delta u + V(x)u = \varepsilon^{2-N-\alpha}\left[I_\alpha * G(x,u)\right]g(x,u)$$

appears in the theory of Bose-Einstein condensation, and is used to describe the finite-range many-body interactions between particles. Here, $G(x,u) = \int_0^u g(x,s)\mathrm{d}s$. For more related results, see, for example, [6,7,12,14,15,17,18,22] and so on.

In particular, the above equation is the so-called Choquard equation, when $N = 3$. For ε =1, $\alpha = 1$, $V(x) \equiv 1$ and $g(x,u) = u$, the autonomous equation

$$-\Delta u + u = \left[I_1 * |u|^2\right]u \quad \text{in } \mathbb{R}^3$$

arises from the quantum theory of a polaron by Pekar [27]. Choquard [20] applied it as an approximation to the Hartree-Fock theory of one-component plasma. In [24], Penrose proposed it as a model of self-gravitating matter. We also mention [38], where the fractional case is treated. Concerning other mathematical and physical background on Choquard problems, see [3,25,28,29,31,33,34] the references therein.

It is well-known that when $N \ge 3$ the Sobolev embedding yields $H^1(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N)$ for all $s \in [2,2^*]$, where $2^* = \frac{2N}{N-2}$. Different from $N \ge 3$, the case $N = 2$ is very special. In such case, the Sobolev exponent 2^* becomes ∞, but $H^1(\mathbb{R}^2) \not\subseteq L^\infty(\mathbb{R}^2)$. Thanks to the Trudinger-Moser inequality below, it provides us a perfect replacement, which was first established by Cao in [8] (also seen in other works [4,5] and reads as follows).

Proposition 1.1 (Cao [8])**.** i) *If* $\beta > 0$ *and* $u \in H^1(\mathbb{R}^2)$, *then*

$$\int_{\mathbb{R}^2}\left(e^{\beta u^2} - 1\right)\mathrm{d}x < \infty;$$

ii) *if* $u \in H^1(\mathbb{R}^2)$, $\|\nabla u\|_2^2 \le 1$, $\|u\|_2 \le M < \infty$, *and* $\beta < 4\pi$, *then there exists a constant* $\mathcal{C}(M,\beta)$, *which depends only on* M *and* β *such that*

$$\int_{\mathbb{R}^2}\left(e^{\beta u^2} - 1\right)\mathrm{d}x \le \mathcal{C}(M,\beta).$$