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Conservation Laws and Exact Solutions to the Modified Hyperbolic Geometric Flow*

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Abstract In this paper, we investigate Lie symmetry group, optimal system, exact solutions and conservation laws of modified hyperbolic geometric flow via Lie symmetry method. Then, conservation laws of modified hyperbolic geometric flow are obtained by applying Ibragimov method.

Keywords The modified hyperbolic geometric flow, exact solution, conservation law

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1. Introduction

Kong and Liu [7] first put forward the hyperbolic geometric flow

$$\frac{\partial^2 g_{ij}}{\partial t^2} + 2R_{ij} + \mathscr{F}(g, \frac{\partial g}{\partial t}) = 0, \tag{1.1}$$

in which g_{ij} is the surface metric, $\mathscr{F}$ is the smooth function of $g, \dfrac{\partial g}{\partial t}$ and R_{ij} is Ricci curvature tensor. Liu [8] discussed the classical global solution to the Cauchy problem of dissipative hyperbolic geometric flow, and discussed that the solution blows up. On the Riemann place, Wang [10] studied the exact solutions, the existence and uniqueness of global solution and the blow up of the solution for the geometrical flows.

Gao and Zhang [2] discussed the group-invariant solutions of the evolution equation of a hyperbolic curve flow by applying the classical Lie symmetry method. They [3] also studied the group invariant solutions of the normal hyperbolic mean curvature flow with dissipation via Lie symmetry method. Gao and Wang [4, 5] studied two different hyperbolic geometry flow equation by Lie symmetry analysis and nonlinear self-adjointness.

A new theorem of conservation laws for arbitrary differential equations is proposed by Ibragimov [6]. Belevtsov and Lukashchuk [1] investigated symmetry group classification by Lie symmetry analysis and constructed the conservation laws of

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the nonlinear fractional diffusion equation with the Riesz potential, which satisfied nonlinear self-adjointness. Zhang, Simbanefayi and Khalique [11] studied the traveling wave solutions and conservation laws of the (2+1)-dimensional Broer-Kaup-Kupershmidt Equation.

Silva [9] studied the nonlinear self-adjointness and conservation laws for the quasilinear 2D second-order evolution equation

$$u_{tt} = Au_{xy}+Bu_xu_y+Cu_{xx}+Du_{yy}+Eu_y+Fu_x+Pu_x^2+Qu_y^2+G+Hu_t+Iu_t^2, \quad (1.2)$$

in which $A, B, C, D, E, F, G, H, I, P, Q$ and R are functions of x, y, t and $u = u(x, y, t)$. He discussed nonlinear self-adjointness and calculated conservation laws on Rieman surfaces for hyperbolic geometric flow equation.

Letting

$$A = \frac{1}{u},\ B = -\frac{1}{u^2},\ G = \lambda u,\ C = D = E = F = P = Q = H = I = 0 \quad (1.3)$$

be in equation (1.2), we obtain

$$u_{tt} = \frac{1}{u}u_{xy} - \frac{1}{u^2}u_xu_y + \lambda u, \quad (1.4)$$

in which λ is an arbitrary constant. Equation (1.3) is known as the modified hyperbolic geometric flow, and it is also given by equation (1.1) with $\mathscr{F} = -\alpha g_{ij}$.

In this paper, we will study the exact solutions and the conservation laws of equation (1.4). First, the Lie point symmetry group for the modified hyperbolic geometric flow is obtained by applying the Lie symmetry method. Second, the Optimal system and exact solutions are discussed. Finally, the conservation laws and nonlocal conservation laws of equation (1.4) are given by applying the Ibragimov method.

2. Lie symmetry group analysis of equation (1.4)

The Lie symmetry of equation (1.4) is generated by the infinitesimal generator

$$V = \rho(x, y, t, u)\frac{\partial}{\partial x} + \sigma(x, y, t, u)\frac{\partial}{\partial y} + \mu(x, y, t, u)\frac{\partial}{\partial t} + \omega(x, y, t, u)\frac{\partial}{\partial u}. \quad (2.1)$$

The second-order prolongation vector field is

$$pr^{(2)}V = \rho\frac{\partial}{\partial x} + \sigma\frac{\partial}{\partial y} + \mu\frac{\partial}{\partial t} + \omega\frac{\partial}{\partial u} + \omega^x\frac{\partial}{\partial u_x} + \omega^y\frac{\partial}{\partial u_y} + \omega^t\frac{\partial}{\partial u_t} + \omega^{xx}\frac{\partial}{\partial u_{xx}} + \omega^{xy}\frac{\partial}{\partial u_{xy}} + \omega^{xt}\frac{\partial}{\partial u_{xt}} + \omega^{yt}\frac{\partial}{\partial u_{yt}} + \omega^{yy}\frac{\partial}{\partial u_{yy}} + \omega^{tt}\frac{\partial}{\partial u_{tt}}.$$

Equation (1.4) remains invariant under an infinitesimal transformation, if and only if V satisfies

$$pr^{(2)}(V)(\Delta)|_{\Delta=0} = 0, \quad (2.2)$$

in which $\Delta = u_{tt} - \frac{1}{u}u_{xy} + \frac{1}{u^2}u_xu_y - \lambda u$, i.e.,

$$\omega(\frac{1}{u^2}u_{xy} - \frac{2}{u^3}u_xu_y - \lambda) + \omega^x\frac{u_y}{u^2} + \omega^y\frac{u_x}{u^2} - \omega^{xy}\frac{1}{u} + \omega^{tt} = 0, \quad (2.3)$$