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Existence and Uniqueness of Smooth Solution for a Four-waves Coupled System*

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Abstract In this paper, we consider a four-waves coupled system which describes the interaction between particles. Based on the uniform bound and the strong convergence property in the lower order norm, local existence and uniqueness of smooth solution are established by a limiting argument. Moreover, we show that the solution exists globally in the two-dimensional case under certain condition on the size for L^2 norm of the initial data.

Keywords Existence and uniqueness, global solution, four-waves coupled system

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1. Introduction

In this paper, we consider the Cauchy problem for a four-waves coupled system which reads

$$\left(\mathrm{i}(\partial_t + v_C\partial_y) + \alpha\Delta\right) A_C = \frac{b^2}{2} n A_C, \tag{1.1}$$

$$\left(\mathrm{i}(\partial_t + v_R\partial_y) + \beta\Delta\right) A_R = \frac{bc}{2} n A_R, \tag{1.2}$$

$$\left(\mathrm{i}\partial_t + \gamma\Delta\right) E = \frac{b}{2} n E, \tag{1.3}$$

$$\left(\partial_t^2 - v_s^2\Delta\right) n = a\Delta\left(|E|^2 + b\,|A_C|^2 + c\,|A_R|^2\right), \tag{1.4}$$

$$(A_C, A_R, E, n)(0) = (a_C, a_R, e, n_0),\ n_t(0) = n_1. \tag{1.5}$$

In this system, $A_C = A_0 + e^{-2iy}A_B$ where A_0 is the incident laser field and A_B is the Brillouin component, A_R is the Raman backscattered wave, E is the electronic-plasma wave and n is the variation of density of ions. Furthermore, $A_C,\ A_R,\ E$ and n are functions of $(x,t) \in \mathbb{R}^d \times \mathbb{R}$ with $A_C,\ A_R$ and E the vector fields such that $A_C,\ A_R,\ E : \mathbb{R}^{d+1} \longrightarrow \mathbb{C}^d$, and with n the scalar field such that $n : \mathbb{R}^{d+1} \longrightarrow \mathbb{R}$. In this paper, we mainly consider the dimension $d = 2, 3$. The coefficients α, β, γ, v_C, v_R, a, b, c and v_s in the above system are real physical constants with $\alpha,\ \beta,\ \gamma > 0$.

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In equation (1.1) and (1.2), y represents the second component of the variable x, namely, $x = (x_1, y, x_3)$ when d=3 and $x = (x_1, y)$ when $d = 2$. With this notation, the Laplacian Δ is defined by

$$\Delta = \partial_{x_1}^2 + \partial_y^2 + \partial_{x_3}^2 \text{ if } d = 3, \quad \Delta = \partial_{x_1}^2 + \partial_y^2 \text{ if } d = 2.$$

In the following arguments of the paper, we often regard y to be x_2 for simplicity. System (1.1)-(1.5) was derived by M. Colin and T. Colin [1] which is a complete set of Zakharov's equations type describing laser-plasma interactions and we have omitted the quasilinear part in this context.

Ignoring the effect of the scattering fields A_C and A_R, system (1.1)-(1.5) is reduced to the classical Zakharov system [18]. Due to its physical importance, the Zakharov system has been studied intensively in mathematics since the works [6,15] and many important developments were obtained in the past decades ([5]). Further, omitting the term n_{tt}, the system is reduced to the cubic Schrödinger equation which has been studied by many researchers, see for example [2,8,12,13] and the references cited therein. For the three-waves (A_C, A_R and E) interacted system, the authors in [9–11] studied the local well-posedness theory.

The work is concerned with the existence and uniqueness of the smooth solution for the four-waves coupled system (1.1)-(1.5). We first introduce a vector-valued function V to rewrite the original system (1.1)-(1.5) as a Hamilton form.

$$(\mathrm{i}(\partial_t + v_C\partial_y) + \alpha\Delta)\, A_C = \frac{b^2}{2} n A_C, \tag{1.6}$$

$$(\mathrm{i}(\partial_t + v_R\partial_y) + \beta\Delta)\, A_R = \frac{bc}{2} n A_R, \tag{1.7}$$

$$(\mathrm{i}\partial_t + \gamma\Delta)\, E = \frac{b}{2} n E, \tag{1.8}$$

$$n_t + \nabla \cdot V = 0, \tag{1.9}$$

$$V_t + v_s^2 \nabla n + a\nabla \left(|E|^2 + b|A_C|^2 + c|A_R|^2\right) = 0, \tag{1.10}$$

$$(A_C, A_R, E, n, V)(0) = (a_C, a_R, e, n_0, V_0). \tag{1.11}$$

Throughout the paper, we denote by $L^p(\mathbb{R}^d)$ the Lebesgue space equipped with the norm

$$\|u\|_{L^p} = \left(\int_{\mathbb{R}^d} |u(x)|^p dx\right)^{\frac{1}{p}} \text{ if } 1 \leq p < +\infty$$

and

$$\|u\|_{L^\infty} = \operatorname{esssup}\{|u(x)|; x \in \mathbb{R}^d\}.$$

For $s \in \mathbb{R}$, $H^s(\mathbb{R}^d)$ denotes the nonhomogeneous Sobolev space defined by

$$H^s(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^d) \mid \|u\|^2_{H^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \left(1 + |\xi|^2\right)^s |\widehat{u}(\xi)|^2 d\xi < +\infty\},$$

where $\widehat{u}(\xi)$ is the Fourier transform of u.

The main results of the paper are stated in the following theorems.

Theorem 1.1. *Letting $d = 2, 3$, assume that a_C, a_R, $e \in H^m(\mathbb{R}^d)$, n_0, $V_0 \in H^{m-1}(\mathbb{R}^d)$, $m \geq 4$ is an integer and $ab > 0$. Then system (1.6)-(1.11) admits a unique solution (A_C, A_R, E, n, V) such that*

$$A_C,\ A_R,\ E \in C\left([0,T); H^m\left(\mathbb{R}^d\right)\right),\ n,\ V \in C\left([0,T); H^{m-1}\left(\mathbb{R}^d\right)\right),$$